Vague and weak convergence of signed measures

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May 27, 2022

Abstract

Sufficient conditions for weak and vague convergence of measures are important for a diverse host of applications. This paper aims to give a comprehensive description of the relationship between the two modes of convergence when the measures are signed, which is largely absent from the literature. Furthermore, when the underlying space is $\mathbb{R}$, we study the relationship between vague convergence of signed measures and the pointwise convergence of their distribution functions.

Keywords: Weak convergence, vague convergence, signed measures

1 Introduction

For positive measures, the relationships between weak convergence, vague convergence and convergence of their distribution functions is well understood; see e.g. Kallenberg [10, 9], Dieudonné and Macdonald, [5] or Daley and Vere-Jones, [4].

These relationships lie at the heart of key results in probability theory such as Karamata's Tauberian theorem (see e.g. Feller [6, XIII.5, Theorem 1]), whose proof relies on the equivalence between the vague convergence of finite positive measures and the pointwise convergence of their distribution functions (at continuity points of the limiting measure).

Motivated by an application in stochastic control, we extended Karamata's theorem to signed measures in Herdegen et al. [13]. This required to study the relationship between vague convergence of signed measures and pointwise convergence of their distribution functions. It turns out that in one direction, the result for positive measures carries over directly, in the other direction, a new condition is needed. Moreover, this investigation resulted in a comprehensive description of the relationship between weak and vague convergence of signed measures, including their Hahn-Jordan decompositions.

1.1 The definition of vague and weak convergence

Throughout this section, let Ω be a Hausdorff space and $\mathscr{B}(\Omega)$ its Borel σ-algebra.

Let $C(\Omega)$ be the space of all continuous $\mathbb{R}$-valued functions on Ω, $C_b(\Omega)$ the subspace of all $f \in C(\Omega)$ such that f is bounded, $C_0(\Omega)$ the subspace of all $f \in C(\Omega)$ such that for any $\varepsilon > 0$, there exists a compact set $K_\varepsilon \in \mathscr{B}(\Omega)$ with $|f| < \varepsilon$ on K_ε^c, and $C_c(\Omega)$ the subspace of all

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$f \in C(\Omega)$ such that f has compact support. Clearly, we have the inclusions $C_c(\Omega) \subseteq C_0(\Omega) \subseteq C_b(\Omega) \subseteq C(\Omega)$.

For a signed measure μ on $(\Omega, \mathscr{B}(\Omega))$, we denote its Hahn-Jordan decomposition by $\mu = \mu^+ - \mu^-$, and its associated variation measure by $|\mu| := \mu^+ + \mu^-$. The *total variation* of a signed measure μ is denoted by $\|\mu\| := |\mu|(\Omega)$, and we say that μ is *finite* if $\|\mu\| < \infty$.

A finite signed measure μ on $(\Omega, \mathscr{B}(\Omega))$ is called a *finite signed Radon measure* if $|\mu|$ is *inner regular*, i.e., for each $A \in \mathscr{B}(\Omega)$,

$$|\mu|(A) = \sup\{|\mu|(K) : K \in \mathscr{B}(\Omega), K \text{ compact}, K \subset A\}.$$

We denote the set of all finite signed Radon measures on $(\Omega, \mathscr{B}(\Omega))$ by $\mathcal{M}(\Omega)$ and the subset of all finite positive Radon measures by $(\mathcal{M}^+(\Omega))$.

We now come to the key definition of this paper.

Definition 1.1. Let Ω be a Hausdorff space. For $\mu \in \mathcal{M}(\Omega)$, define the map $I_\mu : C_b(\Omega) \to \mathbb{R}$ by

$$I_\mu(f) = \int_\Omega f \, \mathrm{d}\mu.$$

We say that a sequence $\{\mu_n\} \subset \mathcal{M}(\Omega)$ converges to $\mu \in \mathcal{M}(\Omega)$

(a) *vaguely* if $I_{\mu_n}(f) \to I_\mu(f)$ for all $f \in C_c(\Omega)$, and we write

$$\operatorname*{v\text{-}lim}_{n\to\infty} \mu_n = \mu;$$

(b) *weakly* if $I_{\mu_n}(f) \to I_\mu(f)$ for all $f \in C_b(\Omega)$,[1] and we write

$$\operatorname*{w\text{-}lim}_{n\to\infty} \mu_n = \mu.$$

Before making some comments on our definition of vague convergence, it is useful to recall the famous Riesz-Markov-Kakutani Representation Theorem; see [7, Theorem 7.17] and [1, Theorem 14.14] for a proof.

Theorem 1.2 (Riesz-Markov-Kakutani Representation Theorem)**.** *Let Ω be a locally compact Hausdorff space.*

(a) *The mapping $\mu \mapsto I_\mu$, where $I_\mu : C_0(\Omega) \to \mathbb{R}$, is an isometric isomorphism from $\mathcal{M}(\Omega)$ to $(C_0(\Omega))^*$.*

(b) *The mapping $\mu \mapsto I_\mu$, where $I_\mu : C_c(\Omega) \to \mathbb{R}$, is a surjective isometry from $\mathcal{M}(\Omega)$ to $(C_c(\Omega))^*$.*

We also note the following straightforward result that sheds light between the relationship of parts (a) and (b) in Theorem 1.2. It follows directly from the Stone-Weierstraß Theorem A.1 and the triangle inequality.

Proposition 1.3. *Let Ω be a locally compact Hausdorff space and $\{\mu_n\} \cup \{\mu\} \subset \mathcal{M}(\Omega)$ with $\sup_{n\in\mathbb{N}} \|\mu_n\| < \infty$. Then*

$$I_{\mu_n}(f) \to I_\mu(f) \text{ for all } f \in C_0(\Omega) \quad \textit{if and only if} \quad I_{\mu_n}(f) \to I_\mu(f) \text{ for all } f \in C_c(\Omega). \tag{1.1}$$

[1] Weak convergence is sometimes referred to as narrow convergence; see [3, Section 8.1].

Now some remarks on our definition of vague convergence are in order, in particular given that one can find a variety of definitions in the extant literature.

Remark 1.4. (a) Our definition of vague convergence is maybe the most common one found in the literature; see e.g. Dieudonné and Macdonald [5, Section XIII.4], Kallenberg [9, Chapter 5] or Klenke, [11, Section 13.2].

(b) Motivated by Theorem 1.2, when Ω a locally compact, vague convergence is defined for test functions in $C_0(\Omega)$ (rather than in $C_c(\Omega)$) by Folland [7, Section 7.3]. However, in light of Proposition 1.3, this stronger definition coincides with our definition if the sequence of measures is uniformly bounded.

(c) When Ω is a Polish space, the vague topology on $\mathcal{M}^+(\Omega)$ (which characterises vague convergence) has alternatively been defined to be generated by the family of mappings π_f : $\mathcal{M}^+(\Omega) \to \mathbb{R}_+$ where the f are nonnegative continuous functions with metric bounded support. This is the approach taken by Kallenberg [10, Section 4.1] and Daley and Vere-Jones [4, Section A2.6]. Basrak and Planinić [2] show that this definition coincides with our definition using using the theory of boundedness due to Hue [8]. Moreover, [2] show explicitly that these vague topologies make $\mathcal{M}^+(\Omega)$ a Polish space in its own right. In particular this latter fact convinced us that our definition is the most natural one.

1.2 Organisation of the paper

The remainder of the paper is organised as follows. Section 2 describes the relationship between vague and weak convergence in $\mathcal{M}(\Omega)$, including the weak and vague convergence of the positive and negative parts in the Hahn-Jordan decomposition. The results are summarised in Table 1. In the special case that $\Omega = \mathbb{R}$, Section 3 studies the relationship between the vague convergence of a family $\{\mu_n\}$ of measures and the pointwise convergence of their distribution functions $\{F_{\mu_n}\}$. Appendix A contains some additional results and proofs.

2 Relationship between vague and weak convergence

We first revisit the direct relationship between weak and vague convergence for signed measures in the case that Ω is a metric space. As a warm-up, we recall that vague convergence allows for a loss of mass in the limit, while weak convergence does not.

Example 2.1. Let μ be the zero measure and $\{\mu_n\} \subset \mathcal{M}(\mathbb{R})$ be such that $\mu_n := \delta_n - \delta_{-n}$, where for $x \in \mathbb{R}$, δ_x denotes the Dirac measure at x. Then $\text{v-lim}_{n\to\infty}\mu_n = \mu$ since for any $f \in C_c(\mathbb{R})$,

$$\lim_{n\to\infty} I_{\mu_n}(f) = \lim_{n\to\infty} (f(n) - f(-n)) = 0 = I_\mu(f).$$

Moreover, it holds that $\lim_{n\to\infty} \mu_n(\Omega) = \mu(\Omega)$, i.e. the *signed mass* is preserved.

Now take $f \in C_b(\mathbb{R})$ such that

$$f(x) = \begin{cases} x & \text{for } x \in (-1, 1), \\ \text{sign}(x) & \text{otherwise,} \end{cases}$$

Thus, we do not have $\text{w-lim}_{n\to\infty} = \mu$ since

$$2 = \lim_{n\to\infty} I_{\mu_n}(f) \neq \lim_{n\to\infty} I_\mu(f) = 0.$$

Intuitively, what goes wrong in Example 2.1 is that mass is "sent to infinity". The precise condition that avoids this is *tightness*.

Definition 2.2. A sequence $\{\mu_n\} \subset \mathcal{M}(\Omega)$ is called *tight* if or any $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset \Omega$ such that

$$\sup_{n\in\mathbb{N}} |\mu_n| (K_\varepsilon^c) \leq \varepsilon. \tag{2.1}$$

Remark 2.3. Since each $\mu \in \mathcal{M}(\Omega)$ is tight by inner regularity of $|\mu|$, we can replace (2.1) by

$$\limsup_{n\to\infty} |\mu_n| (K_\varepsilon^c) \leq \varepsilon. \tag{2.2}$$

Tightness is exactly the condition that lifts vague to weak convergence for positive measure. This remains true for signed measures. The proof of the next result follows from Prohorov's theorem for signed measures, see Theorem A.4.[2]

Proposition 2.4. *Let* $\{\mu_n\} \cup \{\mu\} \subset \mathcal{M}(\Omega)$.

(a) *If* $\text{v-lim}_{n\to\infty} \mu_n = \mu$ *and* $\{\mu_n\}$ *is tight, then* $\text{w-}\lim_{n\to\infty} \mu_n = \mu$.

(b) *If* $\text{w-}\lim_{n\to\infty} \mu_n = \mu$*, then* $\text{v-lim}_{n\to\infty} \mu_n = \mu$*. If in addition* Ω *is separable and complete (i.e., Polish), then* $\{\mu_n\}$ *is tight.*

The heuristic that vague convergence ignores mass "being sent to infinity" leads us to note that vague convergence in $\mathcal{M}(\Omega)$ (without loss of signed mass) can be viewed as weak convergence in $\mathcal{M}(\Omega_\infty)$, where Ω_∞ denotes the one-point compactification of Ω; see Definition A.5. To this end, note that a measure $\mu \in \mathcal{M}(\Omega)$ can be canonically extended to a measure $\mu^\infty \in \mathcal{M}(\Omega_\infty)$ by setting $\mu^\infty(A) := \mu(A)$ for $A \in \mathscr{B}(\Omega)$ and $|\mu^\infty| (\{\infty\}) := 0$. We then have the following result, which follows directly from Proposition 1.3 and Theorem A.6.

Proposition 2.5. *Let* $\{\mu_n\} \cup \{\mu\} \subset \mathcal{M}(\Omega)$ *and suppose that* $\sup_{n\in\mathbb{N}} \|\mu_n\| < \infty$*. Denote by* μ_n^∞ *and* μ^∞ *the canonical extension of* μ_n *and* μ*, respectively. Then* $\text{v-}\lim_{n\to\infty} \mu_n = \mu$ *and* $\mu_n(\Omega) \to \mu(\Omega)$ *if and only if* $\text{w-}\lim_{n\to\infty} \mu_n^\infty = \mu^\infty$.

Remark 2.6. Note that for signed measures, weak convergence in $\mathcal{M}(\Omega_\infty)$ is strictly weaker than weak convergence in $\mathcal{M}(\Omega)$. Indeed, Example 2.1 gives an example of $\{\mu_n\} \cup \{\mu\} \subset \mathcal{M}(\Omega)$ with $\sup_{n\in\mathbb{N}} \|\mu_n\| < \infty$ such that $\text{v-lim}_{n\to\infty} \mu_n = \mu$ and $\mu_n(\Omega) \to \mu(\Omega)$ (and hence $\text{w-}\lim_{n\to\infty} \mu_n^\infty = \mu^\infty$), but $\text{w-lim}_{n\to\infty} \mu_n \neq \mu$.

We next investigate under which conditions vague convergence implies the convergence of the positive and negative parts in the Hahn–Jordan decomposition. The following result shows that the necessary and sufficient extra condition is that no mass is lost on compact sets.

Proposition 2.7. *Let* Ω *be a locally compact metric space and* $\{\mu_n\} \cup \{\mu\} \subset \mathcal{M}(\Omega)$*. Then* $\text{v-lim}_{n\to\infty} \mu_n^\pm = \mu^\pm$ *if and only if* $\text{v-lim}_{n\to\infty} \mu_n = \mu$ *and*

$$\limsup_{n\to\infty} |\mu_n| (K) \leq |\mu| (K). \tag{2.3}$$

for every compact set $K \subset \Omega$.

[2] A direct proof of Proposition 2.4(a) follows also from a generalisation of [9, Lemma 5.20].

Proof. First, suppose that $\text{v-lim}_{n\to\infty} \mu_n^\pm = \mu^\pm$. Then clearly $\text{v-lim}_{n\to\infty} \mu_n = \mu$, and (2.3) is satisfied due to the Portmanteau Theorem in the form of Theorem A.2(b).

Conversely, suppose that $\text{v-lim}_{n\to\infty} \mu_n = \mu$ and (2.3) is satisfied. By Theorem A.3, for every open set $\Theta \subset \Omega$,

$$\liminf_{n\to\infty} |\mu_n|(\Theta) \geq |\mu|(\Theta).$$

Thus, Theorem A.2(b) gives $\text{v-lim}_{n\to\infty} |\mu_n| = |\mu|$. Now $\text{v-lim}_{n\to\infty} {\mu_n}^\pm = \mu^\pm$ follows by noting that

$$\mu_n^+ = \frac{1}{2}(|\mu_n| + \mu_n) \quad \text{and} \quad \mu_n^- = \frac{1}{2}(|\mu_n| - \mu_n).$$

□

Note that Condition (2.3) does not restrict "total mass being lost at infinity". By imposing an additional restriction to mitigate this possibility, we can strengthen Proposition 2.7 to deduce that $\text{w-lim}_{n\to\infty} {\mu_n}^\pm = {\mu_n}^\pm$.

Proposition 2.8. *Let Ω be a locally compact metric space and $\{\mu_n\} \cup \{\mu\} \subset \mathcal{M}(\Omega)$. Then* $\text{w-lim}_{n\to\infty} \mu_n^\pm = \mu^\pm$ *if and only if* $\text{v-lim}_{n\to\infty} \mu_n = \mu$ *and*

$$\limsup_{n\to\infty} \|\mu_n\| \leq \|\mu\| . \tag{2.4}$$

Proof. First, suppose that $\text{w-lim}_{n\to\infty} \mu_n^\pm = \mu^\pm$. Then $\text{w-lim}_{n\to\infty} \mu_n = \mu$ and $\text{w-lim}_{n\to\infty} |\mu_n| = |\mu|$. This implies in particular that $\text{v-lim}_{n\to\infty} \mu_n = \mu$ and

$$\lim_{n\to\infty} \|\mu_n\| = \lim_{n\to\infty} \int_\Omega \mathrm{d}|\mu_n| = \int_\Omega \mathrm{d}|\mu| = \|\mu\| . \tag{2.5}$$

Conversely, suppose that $\text{v-lim}_{n\to\infty} \mu_n = \mu$ and (2.4) is satisfied. By Propositions 2.5 and 2.4, it suffices to show that that (2.3) is satisfied and the sequence $\{\mu_n\}$ is tight.

First, we establish (2.3). Seeking a contradiction, suppose there exists a compact set $K \subset \Omega$ such that

$$\limsup_{n\to\infty} |\mu_n|(K) > |\mu|(K). \tag{2.6}$$

Since K^c is open, it follows from Theorem A.3 that

$$\liminf_{n\to\infty} |\mu_n|(K^c) \geq |\mu|(K^c). \tag{2.7}$$

Adding (2.6) and (2.7), it follows that

$$\limsup_{n\to\infty} \|\mu_n\| = \limsup_{n\to\infty} |\mu_n|(\Omega) \geq \limsup_{n\to\infty} |\mu_n|(K) + \liminf_{n\to\infty} |\mu_n|(K^c) > |\mu|(\Omega) = \|\mu\| ,$$

and we arrive at a contradiction to (2.4).

Next, we show that the sequence $\{\mu_n\}$ is tight. Let $\varepsilon > 0$. By inner regularity of μ, there exists a compact set $K \subset \Omega$ such that $|\mu|(K^c) \leq \varepsilon$. By local compactness of Ω, there exists an open set $K \subset L$ such that its closure $\overline{L} =: K_\varepsilon$ is compact. Using (2.4) and Theorem A.3, we obtain

$$\begin{aligned}\limsup_{n\to\infty} |\mu_n|(K^c) &= \limsup_{n\to\infty} (\|\mu_n\| - |\mu_n|(K)) \leq \limsup_{n\to\infty} (\|\mu_n\| - |\mu_n|(L)) \\ &\leq \|\mu\| - \liminf_{n\to\infty} |\mu_n|(L) \leq \|\mu\| - |\mu|(L) \leq \|\mu\| - |\mu|(K) = \mu(K^c) \leq \varepsilon.\end{aligned}$$

□

Remark 2.9. The easy direction in the proof of Proposition 2.8 extends to Ω being a Hausdorff space.

To summarise, starting from vague convergence $\text{v-lim}_{n\to\infty}\mu_n = \mu$, Proposition 2.4 tells us that we get $\text{w-lim}_{n\to\infty}\mu_n = \mu$ if mass is not "lost at infinity". Proposition 2.7 asserts that if mass is not "lost on compact sets", then we get $\text{v-lim}_{n\to\infty}\mu_n^{\pm} = \mu^{\pm}$. Finally, Proposition 2.8 tells us that if mass is not "lost globally", then we even get $\text{w-lim}_{n\to\infty}\mu_n^{\pm} = \mu^{\pm}$. These results are summarised in Table 1.

Table 1: We assume that Ω is a (complete and separable*, locally compact**) metric space and $\{\mu_n\} \cup \{\mu\} \subset \mathcal{M}(\Omega)$.

Condition(s) A		**Condition(s) B**
$\text{v-lim}_{n\to\infty}\mu_n = \mu$, and $\forall\varepsilon > 0$, $\exists$ compact set K_ε such that $\limsup_{n\to\infty} \lvert\mu_n\rvert (K_\varepsilon^c) \le \varepsilon$	$\Rightarrow$ $\overset{\star}{\Leftarrow}$	$\text{w-lim}_{n\to\infty}\mu_n = \mu$
$\text{v-lim}_{n\to\infty}\mu_n = \mu$, and $\forall$ compact $K \subset \Omega$ $\limsup_{n\to\infty} \lvert\mu_n\rvert (K) \le \lvert\mu\rvert (K)$	$\overset{\star\star}{\Leftrightarrow}$	$\text{v-lim}_{n\to\infty}{\mu_n}^{\pm} = \mu^{\pm}$
$\text{v-lim}_{n\to\infty}\mu_n = \mu$, and $\limsup_{n\to\infty}\lVert\mu_n\rVert \le \lVert\mu\rVert$	$\overset{\star\star}{\Leftrightarrow}$	$\text{w-lim}_{n\to\infty}{\mu_n}^{\pm} = \mu^{\pm}$

3 Vague convergence and convergence of distribution functions

In this section, we study the special case that $\Omega = \mathbb{R}$ and link vague convergence on $\mathbb{R}$ to the pointwise convergence of distribution functions. To this end, we first need to introduce some further pieces of notation.

3.1 Distribution functions

For any open interval $I \subset \mathbb{R}$ and a function $F : I \to [0,\infty]$, we let $\text{Var}_I F$ denote its total variation on I. We let $\text{BV}(I)$ denote the space of all functions of bounded variation on I. If $F \in \text{BV}(I)$, we define the nondecreasing functions $\mathbf{V}_F, F^{\uparrow}, F^{\downarrow} : I \to [0,\infty]$ by $\mathbf{V}_F(x) := \text{Var}_{I\cap(-\infty,x]}F$, $F^{\uparrow}(x) := \frac{1}{2}(\mathbf{V}_F(x) + F(x))$, and $F^{\downarrow}(x) := \frac{1}{2}(\mathbf{V}_F(x) - F(x))$ respectively.

For any $\alpha \in \mathbb{R}$ and $\mu \in \mathcal{M}(\mathbb{R})$, the *distribution function of μ, centred at α,* is the function $F_\mu^{(\alpha)} \in \text{BV}(\mathbb{R})$ defined by

$$F_\mu^{(\alpha)}(x) := \begin{cases} \mu((\alpha, x]) & \text{if } x \ge \alpha, \\ -\mu((x, \alpha]) & \text{if } x < \alpha. \end{cases}$$

Note that $F_\mu^{(\alpha)}$ is right-continuous and for any $a \le b$ with $a, b \in \mathbb{R}$,

$$F_\mu^{(\alpha)}(b) - F_\mu^{(\alpha)}(a) = \mu((a, b]). \tag{3.1}$$

We set $F_\mu := F_\mu^{(0)}$ for convenience.

The relationship (3.1) between distribution functions is bijective, as follows from the following converse statement; for a proof see [12, Theorem 5.13].

Theorem 3.1. *Let $F \in \mathrm{BV}(\mathbb{R})$ be right-continuous. Then there exists a unique $\mu_F \in \mathcal{M}(\mathbb{R})$ such that*

$$\mu_F((a,b]) = F(b) - F(a)$$

for all $a \leq b$ with $a, b \in \mathbb{R}$. Moreover, $|\mu_F| = \mu_{\boldsymbol{V}_F}$.

Let $[-\infty, \infty]$ be the two point compactification of $\mathbb{R}$ to the extended real line. Note that any $\mu \in \mathcal{M}(\mathbb{R})$ can canonically be extended to $\mathcal{M}([-\infty,\infty])$ by setting $|\mu|(\{\pm\infty\}) := 0$. Similarly, F_μ can canonically be extended to $[-\infty, \infty]$ by setting $F_\mu(\pm\infty) := \lim_{x\to\pm\infty} F_\mu(x)$, in which case $F(-\infty) = 0$.

It is particularly interesting to note that, for $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{M}(\mathbb{R})$, we do *not* have that

$$F_{\mu_n} \to F_\mu \text{ for some } \mu \in \mathcal{M}(\mathbb{R}) \;\Leftrightarrow\; \text{w-lim}_{n\to\infty}\mu_n = \mu.$$

Indeed, if $F_{\mu_n} \to F_\mu$ at all continuity points, we can still lose mass at infinity, as shown in Example 2.1. There are even instances where $F_{\mu_n} \to F_\mu$ everywhere, but we do *not* have $\text{v-lim}_{n\to\infty} \mu_n = \mu$. In the following example, this results from $\{F_{\mu_n}\}$ being unbounded on a compact set.

Example 3.2. Let $F_n : \mathbb{R} \to \mathbb{R}$ have support on $[0, 2/n]$ and be linear between the points $\{0, 1/n, 2/n\}$ such that

$$F_n(k/n) = 2^n[k \bmod(2)]$$

for $k \in \{0,1,2\}$. Furthermore, let $\{\mu_n\} \cup \{\mu\} \subset \mathcal{M}(\mathbb{R})$, such that $\mu_n := \lambda_{F_n}$ according to Theorem 3.1, and μ is the zero measure. Then for any $x \in \mathbb{R}$, we have $F_{\mu_n}(x) = F_n(x) \to F_\mu(x)$.

Now take $f \in C_c(\mathbb{R})$ such that

$$f(x) := \begin{cases} (1-x) & \text{for } x \in [0,1], \\ (1+x) & \text{for } x \in [-1,0), \\ 0 & \text{for } x \in [-1,1]^c. \end{cases}$$

Then,

$$I_{\mu_n}(f) = \int_0^{2/n} f(x) F'_{\mu_n}(x)\ \mathrm{d}x = 2^n \left\{ \int_0^{1/n} (1-x)\ \mathrm{d}x - \int_{1/n}^{2/n} (1-x)\ \mathrm{d}x \right\} = \frac{2^n}{n^2}.$$

Thus, $I_{\mu_n}(f) \not\to I_\mu(f)$. See Figure 1 for a clear visualisation.

Furthermore, if $\{\mu_n\} \subset \mathcal{M}(\mathbb{R})$, then $\text{w-}\lim_{n\to\infty} \mu_n = \mu$ does not imply $F_{\mu_n} \to F_\mu$ at continuity points, since mass can be lost locally. This happens when the positive and negative parts of the singular decompositions of $\{\mu_n\}$ cancel in the limit.

Example 3.3. Let $\mu_n := \delta_0 - \delta_{1/n}$, and let μ be the zero measure. Take any $f \in C_b(\mathbb{R})$. Then

$$I_{\mu_n}(f) = f(0) - f(1/n) \to 0 = I_\mu(f).$$

However, we do not have $F_{\mu_n} \to F_\mu$ at all continuity points. Indeed,

$$F_{\mu_n}(x) = \delta_0((0,x]) - \delta_{1/n}((0,x]) = -\mathbb{1}_{\{[1/n,\infty)\}}(x),$$

whence, for any $\varepsilon > 0$,

$$-1 = F_{\mu_n}(\varepsilon) \not\to F_\mu(\varepsilon) = 0.$$

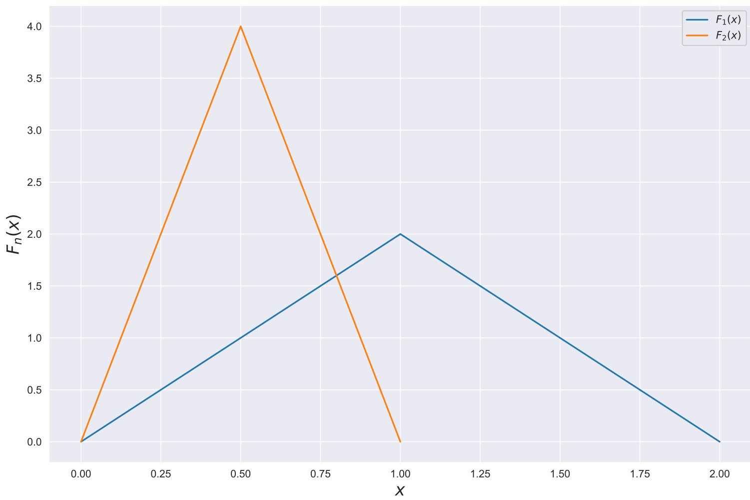

Figure 1: A visualisation of F_1 and F_2 defined in Example 3.2.

Thus, in order to ensure that the distribution functions converge at continuity points, one must ensure that mass is preserved locally, which motivates the following definition.

Definition 3.4. Let $(\Omega, \boldsymbol{\tau})$ be a Hausdorff space and $\{\mu_n\} \subset \mathcal{M}(\Omega)$. We say that $\{\mu_n\}$ has no mass at a point $x \in \Omega$, if for any $\varepsilon > 0$, there exists an open neighbourhood $N_{x,\varepsilon} \in \boldsymbol{\tau}$ of x, such that

$$\limsup_{n\to\infty} |\mu_n| (N_{x,\varepsilon}) \leq \varepsilon.$$

In the case where $\Omega = \mathbb{R}$, we say that $\{\mu_n\}$ has no mass at $+\infty$ (resp. $-\infty$), when the family of canonical extensions of $\{\mu_n\}$ has no mass at $+\infty$ (resp. $-\infty$).

Remark 3.5. Suppose we have a family of tight measures $\{\mu_n\} \subset \mathcal{M}(\mathbb{R})$. By Definition 3.4 we see that requiring $\{\mu_n\}$ to be tight is equivalent to necessitating that the family has no mass at the points $\pm\infty$.

For $\{\mu_n\} \subset \mathcal{M}(\mathbb{R})$, the preceding discussion leads us to a clear characterisation of vague and weak convergence of $\{\mu_n\}$ from the convergence of F_{μ_n}, and vice versa.

Proposition 3.6. *Suppose we have $\{\mu\} \cup \{\mu_n\} \subset \mathcal{M}(\mathbb{R})$ and let $\alpha \in \mathbb{R}$ such that it is not an atom of μ.*

(a) *If $F_{\mu_n}^{(\alpha)}(x) \to F_\mu^{(\alpha)}(x)$ at all continuity points of $F_\mu^{(\alpha)}$ and $\{\mu_n\}$ is bounded on compact sets, then $\text{v-lim}_{n\to\infty} \mu_n = \mu$.*

(b) *If $\text{v-lim}_{n\to\infty} \mu_n = \mu$ and $\{\mu_n\}$ has no mass at all continuity points of $F_\mu^{(\alpha)}$, then $F_{\mu_n}^{(\alpha)} \to F_\mu^{(\alpha)}$ at all continuity points of $F_\mu^{(\alpha)}$.*

Proof. (a) F_μ is continuous except at countably many points, whence $F_{\mu_n}^{(\alpha)} \to F_\mu^{(\alpha)}$ a.e. Take $f \in \mathcal{C} := C^1(\mathbb{R}) \cap C_c(\mathbb{R})$. Then there exists $N \in \mathbb{N}$ such that $\operatorname{supp}(f) \subset K := [-N, N]$, where $\alpha \in K$. Then $\{F_{\mu_n}^{(\alpha)}\}$ is bounded on K since

$$\left|F_{\mu_n}^{(\alpha)}(x)\right| \leq \sup_{n\in\mathbb{N}} |\mu_n|(K) < \infty$$

for all $x \in K$. Hence, using integration by parts and the dominated convergence theorem

$$I_{\mu_n}(f) = -\int_K f'(x)F_{\mu_n}^{(\alpha)}(x)\ \mathrm{d}x \to -\int_K f'(x)F_\mu^{(\alpha)}(x)\ \mathrm{d}x = I_\mu(f).$$

Since $\mathcal{C}$ is a subalgebra of $C_0(\mathbb{R})$, which separates points and vanishes nowhere, it is dense in $C_0(\mathbb{R})$ by Theorem A.1. Take any $f \in C_c(\mathbb{R}) \subset C_0(\mathbb{R})$ and for an arbitrary $\varepsilon > 0$ choose $g \in \mathcal{C}$ such that $\|f - g\|_\infty < \varepsilon$. By taking $M \in \mathbb{N}$ such that $\operatorname{supp}(f) \vee \operatorname{supp}(g) \subset L = [-M, M]$, we see that

$$\begin{aligned}\limsup_{n\to\infty} |I_{\mu_n}(f) - I_\mu(f)| &\leq \limsup_{n\to\infty} \left(|I_{\mu_n}(f-g)| + |I_{\mu_n}(f) - I_\mu(f)| + |I_\mu(f-g)|\right)\\ &\leq \left(\sup_{n\in\mathbb{N}} |\mu_n|(L) + \|\mu\|\right)\varepsilon.\end{aligned}$$

By taking $\varepsilon \downarrow 0$ we may conclude that $\text{v-}\lim_{n\to\infty} \mu_n = \mu$.

(b) Without loss of generality, take any $t > \alpha$ which is a continuity point of $F_\mu^{(\alpha)}$, and take any $\delta > 0$. Define $\rho := \rho_{[\alpha,t],\delta} \in C_c(\mathbb{R})$ such that $\rho_{[\alpha,t],\delta}(x) := (1 - \varepsilon^{-1}\mathrm{d}(x, [\alpha, t]))^+$. Then

$$\begin{aligned}&\limsup_{n\to\infty} \left(\left|F_{\mu_n}^{(\alpha)}(t) - F_\mu^{(\alpha)}(t)\right|\right)\\ \leq\ &\limsup_{n\to\infty} \left(\left|\int (\mathbb{1}_{(\alpha,t]} - \rho)(x)\mu_n(\mathrm{d}x)\right| + \left|\int \rho(x)\mu_n(\mathrm{d}x) - \int \rho(x)\mu(\mathrm{d}x)\right|\right.\\ &\quad \left. + \left|\int (\mathbb{1}_{(\alpha,t]} - \rho)(x)\mu(\mathrm{d}x)\right|\right)\\ \leq\ &\limsup_{n\to\infty} \Big(|\mu_n|((t, t+\delta]) + |\mu_n|((\alpha-\delta, \alpha]) + |\mu|((t, t+\delta]) + |\mu|((\alpha-\delta, \alpha])\Big)\\ \leq\ &\limsup_{n\to\infty} |\mu_n|(B_{2\delta}(t)) + \limsup_{n\to\infty} |\mu_n|(B_{2\delta}(0)) + |\mu|((t, t+\delta]) + |\mu|(\alpha-\delta, \alpha]).\end{aligned}$$

Note that $F_{|\mu|}^{(\alpha)} = \mathbf{V}_{F_\mu^{(\alpha)}}$, and $\mathbf{V}_{F_\mu^{(\alpha)}}$ is continuous at t if and only if $F_\mu^{(\alpha)}$ is continuous at t. Hence,

$$\inf_{\delta>0} |\mu|((t, t+\delta]) = \inf_{\delta>0} \left(F_{|\mu|}^{(\alpha)}(t+\delta) - F_{|\mu|}^{(\alpha)}(t)\right) = 0.$$

Similarly $\inf_{\delta>0} |\mu|((\alpha-\delta, \alpha]) = 0$. Thus, noting that $\{\mu_n\}$ has no mass at t and α, it follows that

$$\begin{aligned}&\limsup_{n\to\infty} \left(\left|F_{\mu_n}^{(\alpha)}(t) - F_\mu^{(\alpha)}(t)\right|\right)\\ \leq\ &\inf_{\delta>0} \left(\limsup_{n\to\infty} |\mu_n|(B_{2\delta}(t)) + \limsup_{n\to\infty} |\mu_n|(B_{2\delta}(0)) + |\mu|((t, t+\delta]) + |\mu|(\alpha-\delta, \alpha])\right)\\ =\ &0.\end{aligned}$$

□

Remark 3.7. Suppose that the measures in Proposition 3.6 are positive. Then $F_{\mu}^{(\alpha)} \to F_{\mu}^{(\alpha)}$ implies that $\{\mu_n\}$ is bounded on compact sets. Indeed, suppose not for some compact set K. Without loss of generality, we may assume that $K \subset [a,b]$ where $\alpha < a < b$, and b is a continuity point of $F_{\mu}^{(\alpha)}$. Then

$$F_{\mu}^{(\alpha)}(b) = \lim_{n\to\infty} F_{\mu_n}^{(\alpha)}(b) = \lim_{n\to\infty} \mu_n((\alpha,b]) \geq \limsup_{n\to\infty} \mu_n(K) = \infty,$$

which is absurd.

Moreover, if $\text{v-lim}_{n\to\infty} \mu_n = \mu$ then the no mass condition of part (b) is automatically satisfied. Indeed, $F_{\mu_n}^{(\alpha)} \to F_{\mu}^{(\alpha)}$ at all continuity points of $F_{\mu}^{(\alpha)}$ by a generalisation of [7, 7.19]. Then, for any continuity point t of $F_{\mu}^{(\alpha)}$, and any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\begin{aligned}\limsup_{n\to\infty} \mu_n(B_\delta(t)) &= \limsup_{n\to\infty} \left(F_{\mu_n}^{(\alpha)}(t+\delta) - F_{\mu_n}^{(\alpha)}(t-\delta)\right) \\ &= F_{\mu}^{(\alpha)}(t+\delta) - F_{\mu}^{(\alpha)}(t-\delta) \leq \varepsilon.\end{aligned}$$

It is worth noting that Proposition 3.6 part (a) does not imply the hypothesis of part (b), as is shown in the following example.

Example 3.8. Suppose we have $\{\mu_n\} \subset \mathcal{M}(\mathbb{R})$ such that $\{\mu_n\}$ is bounded on compact sets and $F_{\mu_n} \to F_\mu$, where μ is the zero measure. It is clear that if the F_{μ_n} have the following three properties, then the family $\{\mu_n\}$ has mass at 0, which is a continuity point of F_μ.

(i) $|F_{\mu_n}(x)| \leq 2^{-n}$ for all $x \in \mathbb{R}$;

(ii) The support of μ_n lies in $[-2^{-n}, 2^{-n}]$;

(iii) ${\mu_n}^{\pm}([-2^{-n}, 0)) = 1/2 = {\mu_n}^{\pm}((0, 2^{-n}])$.

Let $F_n : \mathbb{R} \to \mathbb{R}$ have support on $[-2^{-n}, 2^{-n}]$ and be linear between the points

$$\{\pm 2^{-2n} k : \text{ for } k \in \{0, \dots, 2^n\}\}$$

such that

$$F_n\left(\pm 2^{-2n} k\right) = 2^{-n}[k \bmod(2)]$$

for $k \in \{0, \dots, 2^n\}$. Then $\{\mu_n\} \subset \mathcal{M}(\mathbb{R})$ defined such that $\mu_n := \lambda_{F_n}$ according to Theorem 3.1, have the desired properties. See Figure 2 for a clear visualisation.

Note that Proposition 3.6 allows us to easily show that one can have $\{\mu_n\} \cup \{\mu\} \subset \mathcal{M}(\mathbb{R})$ such that $\text{v-lim}_{n\to\infty} \mu_n = \mu$, $\{\mu_n\}$ has no mass at all points of $\mathbb{R}$, but $\text{v-lim}_{n\to\infty} |\mu_n| \neq |\mu|$.

Example 3.9. Let $F_n : \mathbb{R} \to \mathbb{R}$ have support on $[-1, 1]$ and be linear between the points

$$\{\pm 2^{-n} k : \text{ for } k \in \{0, \dots, 2^n\}\}$$

such that

$$F_n\left(\pm 2^{-n} k\right) = 2^{-n}[k \bmod(2)]$$

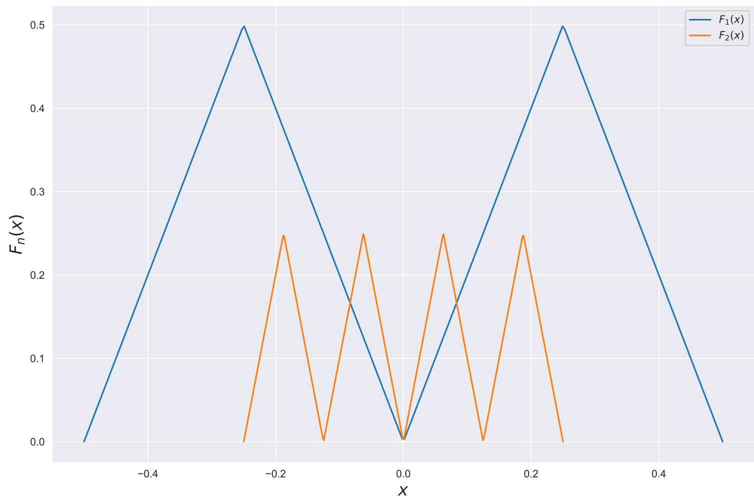

Figure 2: A visualisation of F_1 and F_2 defined in Example 3.8.

for $k \in \{0, \ldots, 2^n\}$. Let $\{\mu_n\} \subset \mathcal{M}(\mathbb{R})$ be characterised by $\{F_n\}$ according to Theorem 3.1. Since $|F_n| \leq 2^{-n}$, it follows from Proposition 3.6 that $\text{v-lim}_{n\to\infty} \mu_n = \mu$, where μ is the zero measure. Moreover, by construction, we have

$$\lim_{\delta \downarrow 0} \sup_{n\in\mathbb{N}} |\mu_n| (B_\delta(x)) = \lim_{\delta\downarrow 0} |\mu_1| (B_\delta(x)) = 0. \tag{3.2}$$

for all $x \in \mathbb{R}$. However,

$$\limsup_{n\to\infty} \|\mu_n\| = 2 \neq 0 = \|\mu\| .$$

See Figure 3 for a clear visualisation.

Corollary 3.10. *Let* $\{\mu_n\}\cup\{\mu\} \subset \mathcal{M}(\mathbb{R})$ *such that* $\limsup_{n\to\infty} \|\mu_n\| \leq \|\mu\|$. *Then* $\text{v-lim}_{n\to\infty} \mu_n = \mu$ *if and only if* $F_{\mu_n} \to F_\mu$ *at all continuity points.*

Proof. Suppose $\text{v-lim}_{n\to\infty} \mu_n = \mu$. Then by Proposition 2.8 it follows that $\text{w-lim}_{n\to\infty} \mu_n^\pm = \mu^\pm$. Hence, by Remark 3.7, $\{\mu_n\}$ has no mass at all continuity points of F_μ. Thus by Proposition 3.6(b), $F_{\mu_n} \to F_\mu$ at all continuity points of F_μ.

Conversely, suppose that $F_{\mu_n} \to F_\mu$ at all continuity points. Since $\sup_{n\in\mathbb{N}} \|\mu_n\| < \infty$, it follows from Proposition 3.6(a) that $\text{v-lim}_{n\to\infty} \mu_n = \mu$. □

Remark 3.11. As a sanity check, one notes that if $\{\mu_n\}\cup\{\mu\} \subset \mathcal{M}(\mathbb{R})$ are probability measures, then Corollary 3.10 and Proposition 2.8 show that $\text{w-lim}_{n\to\infty} \mu_n = \mu$ if and only if $F_{\mu_n} \to F_\mu$ at all continuity points of F_μ. This is often shown as a consequence of Portmanteau's theorem for weak convergence.

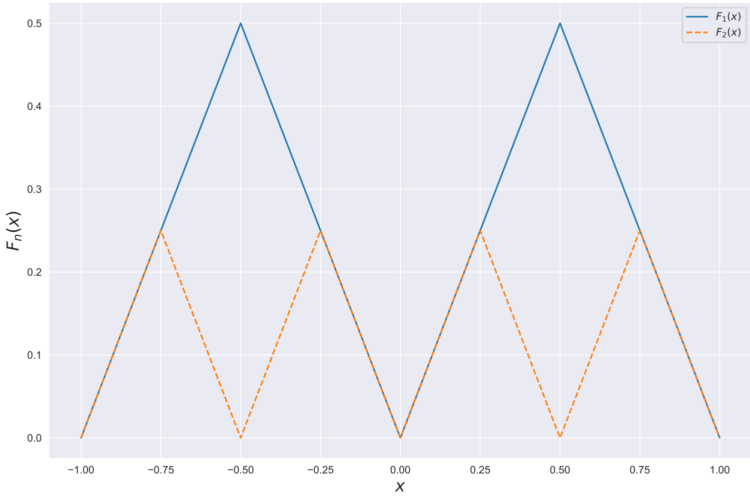

Figure 3: A visualisation of F_1 and F_2 defined in Example 3.9.

A Appendix

A.1 Key results from Functional Analysis and Advanced Measure Theory

In this appendix, we collect some key results from Functional Analysis and Advanced Measure Theory that are used throughout this paper.

First, we recall the famous Stone-Weierstraß Theorem. To this end, recall that a subset $\mathcal{C} \subset C_0(\Omega)$ *vanishes nowhere* if for all $x \in \Omega$, there exists some $f \in \mathcal{C}$ such that $f(x) \neq 0$, and it *separates points* if for each $x, y \in \Omega$ with $x \neq y$, there exists $f \in \mathcal{C}$ such that $f(x) \neq f(y)$.

Theorem A.1 (Stone-Weierstraß Theorem)**.** *Let Ω be a locally compact Hausdorff space and $\mathcal{C}$ be a subalgebra of $C_0(\Omega)$. Then $\mathcal{C}$ is dense in $C_0(\Omega)$ (for the topology of uniform convergence) if and only if it separates points and vanishes nowhere.*

Next, we state a vague version of Portmanteau's Theorem for *positive* measures. For the convenience of the reader, we provide a full proof.

Theorem A.2 (Portmanteau Theorem)**.** *Let Ω be a locally compact metric space and $\{\mu_n\} \cup \{\mu\} \in \mathcal{M}^+(\Omega)$. Then the following are equivalent:*

(a) $\text{v-lim}_{n\to\infty} \mu_n = \mu$.

(b) *For any compact set $K \subset \Omega$,*

$$\limsup_{n\to\infty} \mu_n(K) \leq \mu(K)$$

and for any open set $\Theta \subset \Omega$,

$$\liminf_{n\to\infty} \mu_n(\Theta) \geq \mu(\Theta).$$

(c) *For any set $A \subset \Omega$ such that $A \subset K$ for some compact set K and $\mu(\partial A) = 0$,*

$$\lim_{n\to\infty} \mu_n(A) = \mu(A).$$

Proof. "(a) $\Rightarrow$ (b)" Let $K \subset \Omega$ be compact and for any $\varepsilon > 0$, let $\rho_{K,\varepsilon} := (1 - \varepsilon^{-1}\mathrm{d}(x,K))^+$. Then, by the monotone convergence theorem, we have

$$\limsup_{n\to\infty} \mu_n(K) \leq \inf_{\varepsilon>0} \limsup_{n\to\infty} \int_\Omega \rho_{K,\varepsilon}\ \mathrm{d}\mu_n = \inf_{\varepsilon>0} \int_\Omega \rho_{K,\varepsilon}\ \mathrm{d}\mu = \mu(K).$$

Furthermore, for any open set $\Theta \subset \Omega$, we have

$$\liminf_{n\to\infty} \mu_n(\Theta) \geq \mu(\Theta),$$

directly from Theorem A.3.

"(b) $\Rightarrow$ (c)" Take any $A \subset \Omega$ such that $A \subset K$ for some compact set K and $|\mu|(\partial A) = 0$. Then by our hypotheses

$$\mu(A) \leq \liminf_{n\to\infty} \mu_n(A^\circ) \leq \liminf_{n\to\infty} \mu_n(\overline{A}) \leq \mu(A),$$

whence, $\lim_{n\to\infty} \mu_n(A) = \mu(A)$.

"(c) $\Rightarrow$ (a)" Let $f \in C_c(\Omega)$ and take $\varepsilon > 0$. Set $D := \{y \in \mathbb{R} : \mu(f^{-1}\{y\}) > 0\}$, which must be an at most countable set. Take points $-\|f\|_\infty \leq y_1 \leq \cdots \leq y_{N-1} \leq \|f\|_\infty < y_N$ such that $y_i \in \mathbb{R}\backslash D$ and $y_{i+1} - y_i < \varepsilon$ for $i \in \{1, ..., N-1\}$. Define $\Omega_i := f^{-1}([y_{i-1}, y_i))$ for $i \in \{2, \ldots, N\}$ and note that $\Omega = \bigcup_{i=2}^n \Omega_i$. Since $\partial f^{-1}(D) \subseteq f^{-1}(\partial D)$, it follows that

$$\mu(\partial\Omega_i) \leq \mu(f^{-1}(\{y_{i-1}\})) + \mu(f^{-1}(\{y_i\})) = 0.$$

There exists a unique $j \in \{2, \ldots, N\}$ such that $0 \in [y_{j-1}, y_j)$. Moreover, since $f \in C_c(\Omega)$, there exists a compact set $K \subset \Omega$ such that $f\mathbb{1}_{K^c} = 0$, and $\Omega_i \subset K$ for all $i \in \{2, \ldots, N\}\backslash\{j\}$. By abuse of notation, let us redefine Ω_j as $\Omega_j\backslash K^c$. Thus, it follows from hypothesis that

$$\begin{aligned}
\limsup_{n\to\infty} \int_\Omega f\ \mathrm{d}\mu_n &\leq \limsup_{n\to\infty} \sum_{i=2}^N \mu_n(\Omega_i) y_i \\
&\leq \sum_{i=2}^N \mu(\Omega_i) y_i \\
&\leq \sum_{i=2}^N \mu(\Omega_i) y_{i-1} + \varepsilon \|\mu\| \\
&\leq \int_\Omega f\ \mathrm{d}\mu + \varepsilon \|\mu\|.
\end{aligned}$$

Letting $\varepsilon \downarrow 0$ we get $\limsup_{n\to\infty} \int f\ \mathrm{d}\mu_n \leq \int f\ \mathrm{d}\mu$. By considering $(-f)$ one can obtain $\liminf_{n\to\infty} \int f\ \mathrm{d}\mu_n \geq \int f\ \mathrm{d}\mu$, whence we are done. ☐

The next result is attributed to Varadarajan [14]. For the convenience of the reader, we include a modern proof.

Theorem A.3. *Let Ω be a locally compact normal Hausdorff space. Let $\{\mu_n\} \cup \{\mu\} \subset \mathcal{M}(\Omega)$ and assume that* $\text{v-lim}_{n\to\infty} \mu_n = \mu$. *Then for any open set $\Theta \subset \Omega$,*

$$|\mu|(\Theta) \leq \liminf_{n\to\infty} |\mu_n|(\Theta). \tag{A.1}$$

In particular, $\|\mu\| \leq \liminf_{n\to\infty} \|\mu_n\|$.

Proof. Let $\Theta \subset \Omega$. Let $\varepsilon > 0$ be arbitrary. Since μ is inner regular and Ω is normal and locally compact, by Urysohn's lemma, there exists $f \in C_c(\Omega)$ such that $|f| \leq 1$, $\text{supp}(f) \subset \Theta$ and

$$\int f \, \mathrm{d}\mu \geq |\mu|(\Theta) - \varepsilon.$$

Then by vague convergence of the μ_n,

$$|\mu|(\Theta) - \varepsilon \leq \int f \, \mathrm{d}\mu = \lim_{n\to\infty} \int f \, \mathrm{d}\mu_n \leq \liminf_{n\to\infty} \int |f| \, \mathrm{d}|\mu_n| \leq \liminf_{n\to\infty} |\mu_n|(\Theta)$$

Now the result follows by letting $\varepsilon \downarrow 0$. ☐

Finally, we state a version of Prohorov's theorem for *signed* measures.

Theorem A.4 (Prohorov's Theorem)**.** *Let Ω be a metric space. and $\mathbf{M} \subset \mathcal{M}(\Omega)$ a subset of finite measures.*

(a) *If $\mathbf{M}$ is uniformly bounded and tight, then $\mathbf{M}$ is weakly relatively sequentially compact.*

(b) *If the space Ω is Polish and $\mathbf{M}$ is weakly relatively sequentially compact, then $\mathbf{M}$ is uniformly bounded and tight.*

Proof. (a) Take any $\{\mu_n\} \subset \mathbf{M}$. Since $\mathbf{M}$ is a uniformly bounded and tight family, both $\{\mu_n^+\}$ and $\{\mu_n^-\}$ are uniformly bounded and tight. By [11, Theorem 13.29], it follows that there exists a subsequence $\{n_k\}$ such that $\text{w-lim}_{k\to\infty} \mu_{n_k}^+ = \nu$, for some positive measure $\nu \in \mathcal{M}(\Omega)$. Similarly, there exists a subsequence $\{n_{k_l}\} \subset \{n_k\}$ such that $\text{w-lim}_{l\to\infty} \mu_{n_{k_l}}^- = \eta$, for some positive measure $\mathcal{M}(\Omega)$. Thus it follows that $\text{w-lim}_{l\to\infty} \mu_{n_{k_l}} = (\nu - \eta) \in \mathcal{M}(\Omega)$.

(b) See [3, Theorem 8.6.2]. ☐

A.2 One-point compactification

In this appendix, we recall the one-point compactification of a non-compact locally compact Hausdorff space.

Definition A.5. Let Ω be a non-compact locally compact Hausdorff space with topology $\boldsymbol{\tau}$. Set $\Omega_\infty := \Omega \cup \{\infty\}$, where $\infty \notin \Omega$, and let

$$\boldsymbol{\tau}_\infty := \boldsymbol{\tau} \cup \{\Omega_\infty \backslash K : K \subset \Omega \text{ is compact}\}$$

Then Ω_∞ (with the topology $\boldsymbol{\tau}_\infty$) is called the *one-point* compactification of Ω.

The one-point compactification of a non-compact locally compact Hausdorff space has nice properties; see [7, Proposition 4.36] for a proof.

Theorem A.6. *Let Ω be a non-compact locally compact Hausdorff space. Then Ω_∞ is a compact Hausdorff space and Ω is an open dense subset of Ω_∞. Moreover, $f \in C(\Omega)$ extends continuously to $f_\infty \in C(\Omega_\infty)$ if and only if $f = f_0 + c$ where $f_0 \in C_0(\Omega)$ and c is a constant. In this case, the extension satisfies $f_\infty(\infty) = c$.*

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