Chapter 4
Measures

In this chapter, we shall study the (complex) Banach lattice $M(K)$ consisting of all complex-valued, regular Borel measures on a locally compact space K and, in particular, the positive measures in $M(K)$, which form the cone $M(K)^+$. The Banach space $M(K)$ is isometrically isomorphic to the dual of $C_0(K)$. In §4.2, we shall discuss the linear spaces of discrete measures and of continuous measures on K.

In §4.3, we shall show that a specific quotient of the lattice $M(K)^+$ is a Dedekind complete Boolean ring B such that the Banach space of bounded, continuous functions on the Stone space of B is isometrically isomorphic to the dual space of $M(K)$, and hence to the bidual of $C_0(K)$; this Boolean ring will reappear in §5.4.

We shall also describe, in §4.4, the Banach lattices $L^p(K,\mu)$ and the Boolean algebra $\mathfrak{B}_\mu$ for $\mu \in M(K)^+$ and $1 \leq p \leq \infty$. Important features to be discussed will include consideration of when spaces of the form $C(K)$ are Grothendieck spaces (in §4.5); maximal singular families of measures in $M(K)^+$ (in §4.6), to be used in a later explicit construction of $C_0(K)''$; and the closed subspace $N(K)$ of $M(K)$ consisting of the normal measures (in §4.7). We shall give several examples of spaces with $N(K) = \{0\}$; for example, we shall show in Theorem 4.7.23 that $N(K) = \{0\}$ whenever K is a locally connected, compact space without isolated points. However, we shall show in Theorem 4.7.26 that there is a non-empty, connected, compact space K with $N(K) \neq \{0\}$.

4.1 Measures

Let K be a non-empty, locally compact space. We recall that a *Borel measure* μ on K is a function $\mu : \mathfrak{B}_K \to \mathbb{C}$ such that $\mu(\emptyset) = 0$ and μ is σ-additive, in the sense that

$$\mu(B) = \sum\{\mu(B_n) : n \in \mathbb{N}\}$$

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whenever (B_n) is a sequence of pairwise-disjoint sets in $\mathfrak{B}_K$ with $\bigcup\{B_n : n \in \mathbb{N}\} = B$. Thus a Borel measure on K is just the same as a σ-normal measure on the Boolean algebra $\mathfrak{B}_K$ in the sense of Definition 1.7.12. Further, in the case where $\mu(B) \geq 0$ $(B \in \mathfrak{B}_K)$, the triple $(K, \mathfrak{B}_K, \mu)$ is a measure space.

Definition 4.1.1. Let K be a non-empty, locally compact space, and take a Borel measure μ defined on $\mathfrak{B}_K$. Then

$$|\mu|(B) = \sup \sum_{i=1}^{\infty} |\mu(B_i)| \quad (B \in \mathfrak{B}_K),$$

where the supremum is taken over all partitions of a Borel set B by a countable family $\{B_i : i \in \mathbb{N}\}$ in $\mathfrak{B}_K$. Then $|\mu|$ is the *total variation measure* of μ. The measure μ is *regular* if, for each $B \in \mathfrak{B}_K$ and each $\varepsilon > 0$, there is a compact subset $L \subset B$ and an open set $U \supset B$ with $|\mu|(U \setminus L) < \varepsilon$.

The total variation measure of μ is indeed a Borel measure on K that is regular when μ is regular. On a locally compact space with a countable basis, every Borel measure is regular, but there are compact spaces on which there are Borel measures which are not regular; see [39, §7.1].

Definition 4.1.2. Let K be a non-empty, locally compact space. Then we denote by $M(K)$ the space of complex-valued, regular Borel measures on K, and we set

$$\|\mu\| = |\mu|(K) \quad (\mu \in M(K)).$$

Henceforth, we shall just write 'measure on K' for 'complex-valued, regular Borel measure on K'. The pair $(M(K), \|\cdot\|)$ is a Banach space.

Let L be a closed subspace of K, and take $\mu \in M(L)$. Then we regard μ as an element of $M(K)$ by setting $\mu(B) = \mu(B \cap L)$ $(B \in \mathfrak{B}_K)$. Thus $M(L)$ is a closed subspace of $M(K)$.

The following *Riesz representation theorem* (of F. Riesz) identifies $M(K)$ as the dual space of $C_0(K)$.

Theorem 4.1.3. *Let K be a non-empty, locally compact space. Then the dual space to $C_0(K)$ is identified isometrically with $M(K)$ via the duality specified by*

$$\langle f, \mu \rangle = \int_K f \, \mathrm{d}\mu \quad (f \in C_0(K), \mu \in M(K)).$$

□

In particular, we have the identifications

$$(c_0)''' = (\ell^1)'' = (\ell^\infty)' = C(\beta\mathbb{N})' = M(\beta\mathbb{N}).$$

For details of the Riesz representation theorem, see the recent text of Bogachev [39, §§7.10, 7.11] and the classic texts of Halmos [132] and Rudin [217, Theorem 6.19], for example. The latter two texts were the congenial companions of the authors' distant youths.

Let K be a non-empty, locally compact space. The space of real-valued measures in $M(K)$ is $M_{\mathbb{R}}(K)$. For $\mu, \nu \in M_{\mathbb{R}}(K)$, set

$$\begin{cases} (\mu \vee \nu)(B) = \sup\{\mu(C) + \nu(B \setminus C) : C \in \mathfrak{B}_K, C \subset B\}, \\ (\mu \wedge \nu)(B) = \inf\{\mu(C) + \nu(B \setminus C) : C \in \mathfrak{B}_K, C \subset B\}, \end{cases} \quad (B \in \mathfrak{B}_K). \tag{4.1}$$

Then $M_{\mathbb{R}}(K)$ is a real Banach lattice with respect to the operations $\vee$ and $\wedge$. The definitions in (4.1) agree with those in equation (2.8) when we regard $M_{\mathbb{R}}(K)$ as the dual lattice to $C_{0,\mathbb{R}}(K)$, and so $M_{\mathbb{R}}(K)$ and $M(K)$ are Dedekind complete lattices.

As before, for $\mu \in M_{\mathbb{R}}(K)$, we set $\mu^+ = \mu \vee 0$, $\mu^- = (-\mu) \vee 0$, and

$$|\mu| = \mu^+ + \mu^- = \mu \vee (-\mu),$$

so that $\mu = \mu^+ - \mu^-$, and $|\mu|$ coincides with the total variation measure of Definition 4.1.1; the two measures μ^+ and μ^- are uniquely characterized by the facts that $\mu = \mu^+ - \mu^-$ and $\|\mu\| = \|\mu^+\| + \|\mu^-\|$.

Now take $\mu \in M(K)$. Then we shall write $\Re\mu$ and $\Im\mu$ for the real and imaginary parts of μ, respectively, so that $\mu = \Re\mu + \mathrm{i}\Im\mu$; the *conjugate* of μ is defined to be $\overline{\mu} = \Re\mu - \mathrm{i}\Im\mu$. The measure $|\mu|$ defined in equation (2.5) is indeed the total variation measure of μ defined in Definition 4.1.1. Further, the space $M(K)$, the complexification of $M_{\mathbb{R}}(K)$, is a Banach lattice, and the norm defined by equation (2.7) agrees with that defined in Definition 4.1.2. Clearly the Banach lattice $M(K)$ is an AL-space.

The set of positive measures in $M(K)$ is denoted by $M(K)^+$; this set $M(K)^+$ is weak*-closed in $M(K)$. We note that positive measures correspond to positive linear functionals on $C_0(K)$, in the sense that, for $\mu \in M(K)$, we have $\mu \in M(K)^+$ if and only if $\langle f, \mu \rangle \geq 0 \ (f \in C_0(K)^+)$. We also note that, in the case where K is compact and $\mu \in M(K)$, we have

$$\mu \in M(K)^+ \quad \text{if and only if} \quad \langle 1_K, \mu \rangle = \|\mu\|. \tag{4.2}$$

A measure $\mu \in M(K)^+$ with $\|\mu\| = 1$ is a *probability measure*; the set of these measures is denoted by $P(K)$. In the case where K is compact, $P(K)$ can be identified with the state space $K_{C(K)}$ of the unital C^*-algebra $C(K)$, and $P(K)$ is then clearly a Choquet simplex in the ambient space $(M(K), \sigma(M(K), C(K)))$, and so, as in Example 1.7.15, $\mathrm{Comp}_{P(K)}$ is a complete Boolean algebra.

Let K and L be two non-empty, locally compact spaces, and take $\mu \in M(K)$ and $\nu \in M(L)$. Then there is a unique measure $\mu \otimes \nu \in M(K \times L)$ such that

$$(\mu \otimes \nu)(B \times C) = \mu(B)\nu(C) \quad (B \in \mathfrak{B}_K, C \in \mathfrak{B}_L);$$

$\mu \otimes \nu$ is the *product* of μ and ν. In the case where $\mu \in P(K)$ and $\nu \in P(L)$, we have $\mu \otimes \nu \in P(K \times L)$.

There is one special measure $m \in P(\mathbb{I})$ that we shall use.

Definition 4.1.4. Denote by m the Lebesgue measure on the interval $\mathbb{I} = [0, 1]$.

As well as integrating continuous functions, we can integrate Borel functions against a measure. Recall from Definition 3.3.1 that $B^b(K)$ denotes the space of bounded Borel functions on a locally compact space K.

Definition 4.1.5. Let K be a non-empty, locally compact space. For $f \in B^b(K)$, define $\kappa(f)$ on $M(K)$ by

$$\langle \kappa(f), \mu \rangle = \int_K f \, \mathrm{d}\mu \quad (\mu \in M(K)). \tag{4.3}$$

We see immediately that $\kappa(f) \in M(K)' = C_0(K)''$ and that

$$\mu(B) = \langle \kappa(\chi_B), \mu \rangle \quad (B \in \mathfrak{B}_K, \mu \in M(K)).$$

Indeed, we are regarding each $\mu \in M(K)$ as a continuous linear functional on $B^b(K)$ which extends μ defined on $C_0(K)$; we note that this extension of $\mu \in C_0(K)'$ to $B^b(K)$ is usually not unique.

Let G be a group. Then the identity of G is denoted by e_G. For an element $t \in G$ and subsets S and T of G, we set

$$tS = \{ts : s \in S\}, \quad S^{-1} = \{s^{-1} : s \in S\}, \quad ST = \{st : s \in S, t \in T\}.$$

A *locally compact group* is a group that is also a locally compact topological space such that the group operations are continuous. For example, the Cantor cube $\{0,1\}^\kappa = \mathbb{Z}_2^\kappa$ of weight κ, where κ is an infinite cardinal, is a compact group.

Let G be a locally compact group. Then the Banach space $M(G)$ of all measures on G is a Banach algebra with respect to the *convolution product* $\star$: given measures $\mu, \nu \in M(G)$, we must define $\mu \star \nu$, and we do this by specifying the action of $\mu \star \nu$ on an element $f \in C_0(G)$ and using the Riesz representation theorem. Indeed,

$$\langle f, \mu \star \nu \rangle = \int_G \int_G f(st) \, \mathrm{d}\mu(s) \, \mathrm{d}\nu(t) \quad (f \in C_0(G)).$$

It is standard that $M(G) = (M(G), \star, \|\cdot\|)$ is a unital Banach algebra; the identity is δ_{e_G}. This Banach algebra is called the *measure algebra* of G. For a study of this algebra, see the books [68, 137, 194, 195], and the memoir [72], for example.

Let G be a locally compact group. Then there is a positive measure m_G defined on $\mathfrak{B}_G$ such that $m_G(U) > 0$ for each non-empty, open subset U of G and such that m_G is left-translation invariant, in the sense that $m_G(sB) = m_G(B)$ for each $s \in G$ and $B \in \mathfrak{B}_G$. Such a measure is a *left Haar measure* on G; it is unique up to multiplication by a positive constant. For constructions of this measure, see the classic texts of Hewitt and Ross [137] and Rudin [218].

For example, Haar measure on $(\mathbb{R}, +)$ is the usual Lebesgue measure. Also, set $L = \mathbb{Z}_2^{\mathfrak{c}}$, and let m_L be the product measure on L from the measure on $\{0,1\}$ that gives the value $1/2$ to each of the two points. Then m_L is the Haar measure on L, with $m_L(L) = 1$.

We now return to the spaces $M(K)$. Let K be a non-empty, locally compact space. A measure $\mu \in M(K)$ is *supported* on a Borel subset B of K if $|\mu|(K \setminus B) = 0$. The *support* of a measure $\mu \in M(K)$ is denoted by $\operatorname{supp} \mu$: it is the complement of the union of the open sets U in K such that $|\mu|(U) = 0$, and so is a closed subset of K.

Proposition 4.1.6. *Let K be a non-empty, locally compact space, and suppose that μ is a non-zero measure in $M(K)^+$. Then* $\operatorname{supp} \mu$ *satisfies* CCC. *In the case where K is a compact F-space,* $\operatorname{supp} \mu$ *is Stonean.*

Proof. It follows quickly from the definition of $\operatorname{supp} \mu$ that $\mu(U) > 0$ for each non-empty, open subset U of $\operatorname{supp} \mu$. Thus $\operatorname{supp} \mu$ satisfies CCC. In the case where K is a compact F-space, $\operatorname{supp} \mu$ is Stonean by Proposition 1.5.14. □

A measure $\mu \in M(K)^+$ is *strictly positive* on K if $\mu(U) > 0$ for each non-empty, open subset U of K, equivalently, if $\operatorname{supp} \mu = K$.

We shall use *Hahn's decomposition theorem* and *Lusin's theorem* in the following forms; see [217, Theorems 2.24 and 6.14], for example.

Theorem 4.1.7. *Let K be a non-empty, locally compact space, and take $\mu \in M_{\mathbb{R}}(K)$.*

(i) *There exist Borel subsets P and N of K such that $\{P,N\}$ is a partition of K, such that $\mu(B) \geq 0$ for each Borel subset B of P, and such that $\mu(B) \leq 0$ for each Borel subset B of N.*

(ii) *For each Borel function f on K and each $\varepsilon > 0$, there is a compact subset L of K such that $|\mu|(K \setminus L) < \varepsilon$ and $f \mid L$ is continuous.* □

The partition $\{P,N\}$ in clause (i) of Theorem 4.1.7 is called a *Hahn decomposition of K with respect to μ;* it is unique up to sets of measure zero.

Proposition 4.1.8. *Let K be a non-empty, compact space, and let E be a real-linear subspace of $M_{\mathbb{R}}(K)$ such that*

$$|f|_K = \sup\{|\langle f, \mu\rangle| : \mu \in E_{[1]}\} \quad (f \in C_{\mathbb{R}}(K)).$$

For each non-empty, open subset U of K and each $\varepsilon > 0$, there exists $\mu \in S_E$ with $\mu(U \cap P) > 1 - \varepsilon$, where $\{P,N\}$ is a Hahn decomposition of K with respect to μ.

Proof. Let U be a non-empty, open subset of K, and take $\varepsilon > 0$. Choose $f \in C(K)^+$ with $|f|_K = 1$ and $\operatorname{supp} f \subset U$, and then take $\mu \in S_E$ with $\langle f, \mu\rangle > 1 - \varepsilon$. We see that

$$1 - \varepsilon < \int_K f \, \mathrm{d}\mu = \int_U f \, \mathrm{d}\mu \leq \int_{U \cap P} f \, \mathrm{d}\mu \leq \mu(U \cap P),$$

which gives the result. □

We shall also require the following version of *Choquet's theorem*; we state a general form, which is the *Choquet–Bishop–de Leeuw theorem*; see, for example, [4, §1.4], [104, Theorem 2.10], or [201, §4]. In the case where the specified space

K is metrizable, $\operatorname{ex} K$ is a G_δ-set (by Proposition 2.1.9), and hence a Borel set. As explained in [178, Remark 2.32(c), p. 16], the case of complex scalars is a simple extension of the real case.

Theorem 4.1.9. *Let K be a non-empty, compact, convex subset of a locally convex space E over $\mathbb{R}$ or $\mathbb{C}$, and let $x_0 \in K$. Then there exists $\mu \in P(K)$ such that*

$$\langle x_0, \lambda \rangle = \int_K \lambda \,\mathrm{d}\mu = \int_K \langle x, \lambda \rangle \,\mathrm{d}\mu(x) \quad (\lambda \in E') \tag{4.4}$$

and such that μ vanishes on every Baire subset and on every G_δ-subset of K which is disjoint from $\operatorname{ex} K$. In the case where K is metrizable, $\mu(\operatorname{ex} K) = 1$. □

In the above setting, x_0 is termed the *resultant* or *barycentre* of the measure μ.

We shall use the following known application of the Choquet–Bishop–de Leeuw theorem. It is given in [104, Theorem 2.18]; the proof here is somewhat shorter.

Theorem 4.1.10. *Let E be a normed space, and let K be a weak*-compact, convex subset of E'. Suppose that D is a countable, $\|\cdot\|$-dense subset of $\operatorname{ex} K$. Then K is the $\|\cdot\|$-closure of the convex hull of D, and so K is $\|\cdot\|$-separable.*

Proof. The result is trivial when D is finite, and so we may suppose that D is infinite, say $D = \{\lambda_i : i \in \mathbb{N}\}$. Fix $\varepsilon > 0$, and, for each $i \in \mathbb{N}$, set

$$K_i = \{\lambda \in K : \|\lambda - \lambda_i\| \le \varepsilon\},$$

so that K_i is a weak*-compact subspace of E' and $\operatorname{ex} K \subset \bigcup\{K_i : i \in \mathbb{N}\} \subset K$. Take $\lambda_0 \in K$. By Theorem 4.1.9, there exists $\mu_0 \in P(K)$ such that

$$\langle x, \lambda_0 \rangle = \int_K \langle x, \lambda \rangle \,\mathrm{d}\mu_0(\lambda) \quad (x \in E)$$

and such that μ_0 vanishes on each G_δ-subset of K that is disjoint from $\operatorname{ex} K$. Clearly $\bigcap\{K \setminus K_i : i \in \mathbb{N}\}$ is such a G_δ-set, and so $\mu_0(\bigcup\{K_i : i \in \mathbb{N}\}) = 1$.

Choose pairwise-disjoint Borel sets B_i for $i \in \mathbb{N}$ such that $B_i \subset K_i \ (i \in \mathbb{N})$ and $\bigcup_{i \in \mathbb{N}} B_i = \bigcup_{i \in \mathbb{N}} K_i$, and set $\alpha_i = \mu_0(B_i) \in \mathbb{I} \ (i \in \mathbb{N})$, so that $\sum_{i=1}^\infty \alpha_i = 1$. Next set

$$\Lambda = \sum_{i=1}^\infty \alpha_i \lambda_i \in \overline{\operatorname{co}}\, D.$$

Take $x \in E_{[1]}$. For each $i \in \mathbb{N}$ and $\lambda \in B_i$, we have $|\langle x, \lambda_i \rangle - \langle x, \lambda \rangle| < \varepsilon$, and so

$$\left| \langle x, \alpha_i \lambda_i \rangle - \int_{B_i} \langle x, \lambda \rangle \,\mathrm{d}\mu_0(\lambda) \right| \le \alpha_i \varepsilon.$$

It follows that $|\langle x, \Lambda \rangle - \langle x, \lambda_0 \rangle| \le \varepsilon$, and hence $\|\Lambda - \lambda_0\| \le \varepsilon$. Thus $K = \overline{\operatorname{co}}\, D$. □

Definition 4.1.11. Let K be a non-empty, compact, convex subset of a locally convex space, and suppose that $\mu, \nu \in M(K)^+$. Then

$$\mu \approx \nu \quad \text{if} \quad \langle h, \mu \rangle = \langle h, \nu \rangle \tag{4.5}$$

for each affine function $h \in C_{\mathbb{R}}(K)$, and

$$\mu \prec \nu \quad \text{if} \quad \langle h, \mu \rangle \leq \langle h, \nu \rangle \tag{4.6}$$

for each convex function $h \in C_{\mathbb{R}}(K)$.

Let K be a non-empty, compact, convex subset of a locally convex space. The relation $\prec$ is a partial order on $M(K)^+$; a measure $\mu \in M(K)^+$ is *maximal* if it is maximal in the partially ordered set $(M(K)^+, \prec)$. It is shown in [201, Lemma 4.1] that, for each $\nu \in M(K)^+$, there is a maximal measure $\mu \in M(K)^+$ with $\nu \prec \mu$.

The following result combines Propositions 3.1 and 10.3 of [201] and the *Choquet–Meyer theorem* from [201, p. 56]. Recall that a Choquet simplex was defined within Example 1.7.15.

Theorem 4.1.12. *Let K be a non-empty, compact, convex subset of a locally convex space. Suppose that $\mu \in P(K)$ is such that* $\operatorname{supp}\mu \subset \operatorname{ex}K$*. Then μ is a maximal measure on K. Suppose further that K is a Choquet simplex. Then, for each $x \in K$, there is a unique maximal measure μ such that $\mu \approx \varepsilon_x$.* □

Proposition 4.1.13. *Let K be a non-empty, locally compact space. Suppose that (μ_α) is a net in $M(K)$ which converges to $\mu \in M(K)$ in the weak* topology $\sigma(M(K), C_0(K))$. Then*

$$|\mu|(U) \leq \liminf_\alpha |\mu_\alpha|(U)$$

for each open set U in K. In particular, $\|\mu\| \leq \liminf_\alpha \|\mu_\alpha\|$.

Further, the following maps from $(M(K), \sigma(M(K), C_0(K)))$ to $\mathbb{R}$ are lower semi-continuous: $\mu \mapsto |\mu|(U)$, for each fixed open subset U of K; $\mu \mapsto \int_K g \, \mathrm{d}|\mu|$, for each fixed $g \in C_0(K)^+$; $\mu \mapsto \|\mu\|$.

Proof. Let U be a non-empty, open set in K, and choose $\varepsilon > 0$. Then there exists $f \in C_{00}(K)_{[1]}$ such that $|f| \leq \chi_U$ and $|\int_K f \, \mathrm{d}\mu| > |\mu|(U) - \varepsilon$. For each α, we have

$$|\mu_\alpha|(U) = \int_K \chi_U \, \mathrm{d}|\mu_\alpha| \geq \int_K |f| \, \mathrm{d}|\mu_\alpha| \geq \left| \int_K f \, \mathrm{d}\mu_\alpha \right|,$$

and so

$$\liminf_\alpha |\mu_\alpha|(U) \geq \lim_\alpha \left| \int_K f \, \mathrm{d}\mu_\alpha \right| = \left| \int_K f \, \mathrm{d}\mu \right| > |\mu|(U) - \varepsilon,$$

giving the main result. The remainder is clear. □

Note that the map $\mu \mapsto |\mu|$ on $M(K)$ is not always weak*-weak*-continuous. For example, for $n \in \mathbb{N}$, set

$$s_n(t) = \sin(nt) \quad (t \in \mathbb{I}),$$

and regard (s_n) as a sequence in $L^1(\mathbb{I}) \subset M(\mathbb{I})$. Then (s_n) converges weakly to 0 in $L^1(\mathbb{I})$. To see this, let J be a subinterval of $\mathbb{I}$. Then $\int_J s_n(t)\,\mathrm{d}t \to 0$ as $n \to \infty$, and so $\int_{\mathbb{I}} f(t)s_n(t)\,\mathrm{d}t \to 0$ as $n \to \infty$ whenever f is a finite linear combination of characteristic functions of intervals. Since each $f \in L^\infty(\mathbb{I})$ is the limit in $\|\cdot\|_1$ of such functions, $\int_{\mathbb{I}} f(t)s_n(t)\,\mathrm{d}t \to 0$ as $n \to \infty$ for each $f \in L^\infty(\mathbb{I})$. In particular, (s_n) converges weak* to 0 in $M(\mathbb{I})$. But of course $(|s_n|)$ does not converge weak* to 0.

Let K and L be two non-empty, compact spaces, and again suppose that $\eta : K \to L$ is a continuous surjection. For $\mu \in M(K)$, there is a measure $\nu = (\eta^\circ)'(\mu) \in M(L)$, called the *image* of μ, such that

$$\int_K \eta^\circ(f)(x)\,\mathrm{d}\mu(x) = \int_K (f \circ \eta)(x)\,\mathrm{d}\mu(x) = \int_L f(y)\,\mathrm{d}\nu(y) \quad (f \in C_{00}(L)).$$

It is proved in [132, Theorem 39 (C)] and [138, Theorem (12.46(i))] that

$$\nu(B) = \mu(\eta^{-1}(B)) = \int_K (\chi_B \circ \eta)(x)\,\mathrm{d}\mu(x) \quad (B \in \mathfrak{B}_L). \tag{4.7}$$

We write $\eta[\mu]$ for the image measure ν, so that $\eta[\mu] \in M(L)$; in the case where $\mu \in P(K)$, we have $\eta[\mu] \in P(L)$. The following three results are taken from [206]; see Theorem 4.7.26 for our application of the results.

Proposition 4.1.14. *Let L be a non-empty, connected, compact space. Suppose that $\nu \in P(L)$ is a strictly positive measure and that F is a closed subset of L such that $\nu(F) > 0$. Then there are a non-empty, connected, compact space K containing L as a closed subspace, a strictly positive measure $\mu \in P(K)$, and a continuous surjection $\eta : K \to L$ such that $\eta[\mu] = \nu$ and $\mathrm{int}_K \eta^{-1}(F) \neq \emptyset$.*

Proof. Let $F_0 = \mathrm{supp}\,(\nu \mid F)$, so that

$$F_0 = F \setminus \bigcup \{U : U \text{ open in } L,\ \nu(F \cap U) = 0\}.$$

Set $K = (F_0 \times \mathbb{I}) \cup (L \times \{0\})$, so that K is a non-empty, connected, compact subspace of $F \times \mathbb{I}$. The map η is defined by $\eta(x,t) = x$ $((x,t) \in K)$, so that $\eta : K \to L$ is a continuous surjection. The set $\eta^{-1}(F)$ contains $F_0 \times (0,1]$, and the latter is a non-empty, open subset of K, and so $\mathrm{int}_K \eta^{-1}(F) \neq \emptyset$.

Let $C \in \mathfrak{B}_K$, and define $\mu(C)$ by setting

$$\mu(C) = \nu(C \cap (L \setminus F_0)) + (\nu \otimes m)((F_0 \times \mathbb{I}) \cap C),$$

where we recall that m denotes Lebesgue measure on $\mathbb{I}$. Then it is clear that $\mu \in P(K)$ and that μ is strictly positive. Further, $\mu(\eta^{-1}(B)) = \nu(B)$ $(B \in \mathfrak{B}_L)$, and so $\eta[\mu] = \nu$. □

The notion of an inverse limit of an inverse system of compact spaces arose in Definition 1.4.31.

Let κ be an ordinal. An *inverse system with measures* is an inverse system of compact spaces $(K_\alpha, \pi_\alpha^\beta : 0 \le \alpha \le \beta < \kappa)$, together with measures $\mu_\alpha \in P(K_\alpha)$ for each α with $0 \le \alpha < \kappa$ such that $\pi_\alpha^\beta[\mu_\beta] = \mu_\alpha$ for $0 \le \alpha \le \beta < \kappa$; such a system is denoted by

$$(K_\alpha, \mu_\alpha, \pi_\alpha^\beta : 0 \le \alpha \le \beta < \kappa).$$

Proposition 4.1.15. *Let κ be an ordinal, let $(K_\alpha, \mu_\alpha, \pi_\alpha^\beta : 0 \le \alpha \le \beta < \kappa)$ be an inverse system of compact spaces with measures, and take (K, π_α) to be the inverse limit of $(K_\alpha, \pi_\alpha^\beta : 0 \le \alpha \le \beta < \kappa)$. Then there is a unique measure $\mu \in P(K)$ such that $\pi_\alpha[\mu] = \mu_\alpha$ for $0 \le \alpha < \kappa$. In the case where each μ_α is strictly positive, the measure μ is strictly positive.*

Proof. For each ordinal α with $0 \le \alpha < \kappa$, the map π_α° identifies $C(K_\alpha)$ with a unital, self-adjoint, closed subalgebra, say A_α, of $C(K)$. Set $A = \bigcup\{A_\alpha : 0 \le \alpha < \kappa\}$. Then A separates the points of K, and so, by the Stone–Weierstrass theorem, Theorem 1.4.26(ii), A is dense in $(C(K), |\cdot|_K)$. Set

$$\lambda(f) = \int_{K_\alpha} f \, d\mu_\alpha \quad (f \in A_\alpha).$$

Since $\pi_\alpha^\beta[\mu_\beta] = \mu_\alpha$ for $0 \le \alpha \le \beta < \kappa$, the value of $\lambda(f)$ is independent of the choice of α. It is clear that λ is a positive, continuous linear functional on $(A, |\cdot|_K)$ with $\|\lambda\| = 1$, and so λ extends to a positive, continuous linear functional on $(C(K), |\cdot|_K)$ with $\|\lambda\| = 1$. By the Riesz representation theorem, there exists $\mu \in P(K)$ such that $\lambda(f) = \langle f, \mu \rangle$ $(f \in C(K))$. The measure μ has the required properties. ◻

Theorem 4.1.16. *Let L be a non-empty, connected, compact space, and suppose that $\nu \in P(L)$ is a strictly positive measure. Then there are a non-empty, connected, compact space $L^\#$, a strictly positive measure $\mu^\# \in P(L^\#)$, and a continuous surjection $\eta^\# : L^\# \to L$ such that $\eta^\#[\mu^\#] = \nu$ and $\mathrm{int}_{L^\#}(\eta^\#)^{-1}(Z) \ne \emptyset$ for each $Z \in \mathbf{Z}(L)$ with $\nu(Z) > 0$.*

Proof. Let $\{Z_\alpha : 0 \le \alpha < \kappa\}$ be an enumeration of the sets $Z \in \mathbf{Z}(L)$ with $\nu(Z) > 0$, where κ is a cardinal. We shall define inductively an inverse system with strictly positive measures

$$(K_\alpha, \mu_\alpha, \pi_\alpha^\beta : 0 \le \alpha \le \beta < \kappa)$$

such that $K_0 = L$ and $\mu_0 = \nu$.

In the case where $0 \le \gamma < \kappa$ is such that $(K_\alpha, \mu_\alpha, \pi_\alpha^\beta : 0 \le \alpha \le \beta \le \gamma)$ is an inverse system with non-empty, connected, compact spaces K_α and strictly positive measures μ_α (for $0 \le \alpha \le \gamma$), we define $K_{\gamma+1}$ and $\mu_{\gamma+1}$ by applying Proposition 4.1.14 with $L = K_\gamma$, with $\nu = \mu_\gamma$, and with $F = (\pi_0^\gamma)^{-1}(Z_\gamma)$ (and also defining the maps

$\pi_\alpha^{\gamma+1}$ to be $\eta \circ \pi_\alpha^\gamma$ for $0 \le \alpha \le \gamma$, where η arises in Proposition 4.1.14, and $\pi_{\gamma+1}^{\gamma+1}$ to be the identity on $K_{\gamma+1}$).

In the case where $0 \le \gamma \le \kappa$, γ is a limit ordinal, and $(K_\alpha, \mu_\alpha, \pi_\alpha^\beta : 0 \le \alpha \le \beta < \gamma)$ is an inverse system with non-empty, connected, compact spaces K_α and strictly positive measures μ_α, we define $(K_\gamma, \pi_\alpha^\gamma : 0 \le \alpha < \gamma)$ to be the inverse limit of $(K_\alpha, \pi_\alpha^\beta : 0 \le \alpha \le \beta < \gamma)$ (and take π_α^γ to be the continuous surjections that arise in Theorem 1.4.32), so that K_γ is compact and connected by Theorem 1.4.32; we take $\mu_\gamma \in P(K_\gamma)$ to be the measure specified in Proposition 4.1.15. In the special case in which $\gamma = \kappa$, we set $L^\# = K_\gamma$, $\mu^\# = \mu_\gamma \in P(L^\#)$, and $\eta^\# = \pi_0^\kappa : L^\# \to L$, so that $\eta^\#[\mu^\#] = \nu$. Then $L^\#$, $\mu^\#$ and $\eta^\#$ have the required properties.

Now suppose that $Z \in \mathbf{Z}(L)$ with $\nu(Z) > 0$. Then $Z = Z_\alpha$ for some $\alpha < \kappa$. The interior of the set

$$(\pi_0^{\alpha+1})^{-1}(Z_\alpha) = \left((\pi_\alpha^{\alpha+1})^{-1} \circ (\pi_0^\alpha)^{-1}\right)(Z_\alpha)$$

is non-empty by the basic construction of Proposition 4.1.14, and so we see that $\mathrm{int}_{L^\#}(\eta^\#)^{-1}(Z) = \mathrm{int}_{L^\#}(\eta^\#)^{-1}(Z_\alpha) \neq \emptyset$, as required. □

In the case where $L = \mathbb{I}$ and $\nu = m$, we see that $|\{Z \in \mathbf{Z}(L) : \nu(Z) > 0\}| = \mathfrak{c}$, and so $\kappa = \mathfrak{c}$ in the above proof. It follows by an easy induction that $w(L^\#) = \mathfrak{c}$.

4.2 Discrete and continuous measures

We now introduce discrete, continuous, singular, and absolutely continuous measures.

Definition 4.2.1. Let K be a non-empty, locally compact space. The measures μ for which every set A with $|\mu|(A) > 0$ contains a point x with $|\mu|(\{x\}) > 0$ are the *discrete* measures, and the measures μ such that $\mu(\{x\}) = 0$ for each $x \in K$ are the *continuous* measures.

Let K be a non-empty, locally compact space. The sets of discrete and continuous measures on K are denoted by $M_d(K)$ and $M_c(K)$, respectively; they are closed linear subspaces of $M(K)$ and

$$M(K) = M_d(K) \oplus_1 M_c(K). \tag{4.8}$$

Further, both $M_d(K)$ and $M_c(K)$ are closed $C_0(K)$-submodules of $M(K)$, both are lattice ideals in $M(K)$, and it is standard that $M_d(K)$ is $\sigma(M(K), C_0(K))$-dense in $M(K)$; see Corollary 4.4.16. The point mass at $x \in K$ is denoted by δ_x, so that $\delta_x \in M_d(K)$. Indeed, $M_d(K) = \ell^1(K)$ when we identify the measure δ_x with the function $\chi_{\{x\}}$ for $x \in K$. The measure m on $\mathbb{I}$ is continuous. We set

$$P_d(K) = P(K) \cap M_d(K) \quad \text{and} \quad P_c(K) = P(K) \cap M_c(K).$$

Proposition 4.2.2. *Let K be a non-empty, locally compact space that contains a countable, dense subset Q, and suppose that $\mu \in M_c(K)^+$. Then K contains a dense G_δ-subset D such that $Q \subset D$ and $\mu(D) = 0$.*

Proof. Set $Q = \{x_n : n \in \mathbb{N}\}$. Since the measure μ is continuous, it follows that, for each $k, n \in \mathbb{N}$, there is an open neighbourhood $U_{k,n}$ of x_n such that $\mu(U_{k,n}) < 1/2^n k$. Set

$$U_k = \bigcup \{U_{k,n} : n \in \mathbb{N}\} \quad (k \in \mathbb{N}).$$

Then each U_k is an open subset of K with $\mu(U_k) < 1/k$. The set $D := \bigcap \{U_k : k \in \mathbb{N}\}$ is a G_δ-subset of K; it is dense in K because it contains $\{x_n : n \in \mathbb{N}\}$, and clearly $\mu(D) = 0$. □

Proposition 4.2.3. *Let K be an uncountable, compact, metrizable space. Then we have $|M(K)| = \mathfrak{c}$.*

Proof. By Proposition 1.4.14, $|K| = \mathfrak{c}$, and so $|M(K)| \geq |M_d(K)| \geq \mathfrak{c}$.

The topological space K has a countable base; we may suppose that this base is closed under finite unions. Each open set in K is a countable, increasing union of members of the base, and so each $\mu \in M(K)$ is determined by its values on the sets of this base. Hence $|M(K)| \leq \mathfrak{c}$. □

Definition 4.2.4. Let K be a non-empty, locally compact space, and suppose that $\mu, \nu \in M(K)$. Then $\mu \perp \nu$ if μ and ν are *mutually singular*, in the sense that there exists $B \in \mathfrak{B}_K$ with $|\mu|(B) = 0$ and $|\nu|(K \setminus B) = 0$, and $\mu \ll \nu$ if $|\mu|$ is *absolutely continuous* with respect to $|\nu|$, in the sense that $|\mu|(B) = 0$ whenever $B \in \mathfrak{B}_K$ and $|\nu|(B) = 0$.

For $\mu, \nu \in M(K)$, set

$$\mu \sim \nu \quad \text{if} \quad \mu \ll \nu \quad \text{and} \quad \nu \ll \mu .$$

We recall that $\mu \ll \nu$ if and only if, for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $|\mu(B)| < \varepsilon$ whenever $B \in \mathfrak{B}_K$ and $|\nu|(B) < \delta$. Suppose that $\mu, \nu \in M(K)$ with $\mu \ll \nu$. Then $\operatorname{supp} \mu \subset \operatorname{supp} \nu$.

It is easy to check that $\sim$ is an equivalence relation on the space $M(K)$. Clearly $\mu \sim |\mu|$ for each $\mu \in M(K)$.

It follows from the Hahn decomposition theorem that each $\mu \in M(K)$ has a *Jordan decomposition*:

$$\mu = \mu_1 - \mu_2 + \mathrm{i}(\mu_3 - \mu_4), \tag{4.9}$$

where $\mu_1 = (\Re\mu)^+$, $\mu_2 = (\Re\mu)^-$, $\mu_3 = (\Im\mu)^+$, and $\mu_4 = (\Im\mu)^-$. Note that $\mu_1, \mu_2, \mu_3, \mu_4 \in M(K)^+$ and $\mu_j \ll \mu$ for $j = 1, 2, 3, 4$.

The following inequality, which follows easily, will be useful. Let K be a non-empty, locally compact space, and take $\mu \in M(K)$. Then, for each $B \in \mathfrak{B}_K$, we have

$$|\mu|(B) \leq 4 \sup \{|\mu(C)| : C \in \mathfrak{B}_K, C \subset B\} . \tag{4.10}$$

The following two results are clear.

Proposition 4.2.5. *Let K be a non-empty, locally compact space, and suppose that $\mu, \nu \in M(K)$. Then $\mu \perp \nu$ if and only if*

$$\|\mu + \nu\| = \|\mu - \nu\| = \|\mu\| + \|\nu\| .$$

$\square$

Corollary 4.2.6. *Let K and L be non-empty, locally compact spaces. Suppose that E is a linear subspace of $M(K)$ and that $T : E \to M(L)$ is a linear isometry. Take measures $\mu, \nu \in E$. Then $T\mu \perp T\nu$ if and only if $\mu \perp \nu$.* $\square$

Proposition 4.2.7. *Let K be a non-empty, locally compact space, and suppose that $\mu \in M(K)$. Then μ is continuous if and only if, for each $\varepsilon > 0$, there exist $n \in \mathbb{N}$ and $\mu_1, \dots, \mu_n \in M(K)$ with*

$$\mu = \mu_1 + \cdots + \mu_n ,$$

with $\mu_i \perp \mu_j \ (i, j \in \mathbb{N}_n, i \neq j)$, and with $\|\mu_i\| < \varepsilon \ (i \in \mathbb{N}_n)$.

Proof. Suppose that $\mu \in M_c(K)$, and take $\varepsilon > 0$. Then there is a compact subset L of K such that $|\mu|(K \setminus L) < \varepsilon$. Each point $x \in L$ has an open neighbourhood U_x with $|\mu|(U_x) < \varepsilon$, and the union, say $\bigcup\{U_j : j \in \mathbb{N}_n\}$, of finitely many of these neighbourhoods contains L. Set $V_1 = U_1$ and $V_j = U_j \setminus (U_1 \cup \cdots \cup U_{j-1})$ for $j = 2, \dots, n$. Then set $\mu_0 = \mu \mid (K \setminus L)$ and $\mu_j = \mu \mid (V_j \cap L) \ (j \in \mathbb{N}_n)$. We see that $\mu_0, \mu_1, \dots, \mu_n \in M(K)$, and they have the required properties (after re-labelling).

The converse is immediate. $\square$

Corollary 4.2.8. *Let K and L be non-empty, locally compact spaces, and suppose that $T : M(K) \to M(L)$ is a linear isometry. Then $T\mu \in M_c(L)$ whenever $\mu \in M_c(K)$.*

Proof. Take $\mu \in M_c(K)$ and $\varepsilon > 0$. Then there exist $n \in \mathbb{N}$ and $\mu_1, \dots, \mu_n \in M(K)$ with $\mu = \mu_1 + \cdots + \mu_n$, with $\mu_i \perp \mu_j \ (i, j \in \mathbb{N}_n, i \neq j)$, and with $\|\mu_i\| < \varepsilon \ (i \in \mathbb{N}_n)$. Then $T\mu = T\mu_1 + \cdots + T\mu_n$, with $T\mu_i \perp T\mu_j \ (i, j \in \mathbb{N}_n, i \neq j)$ by Corollary 4.2.6 and with $\|T\mu_i\| < \varepsilon \ (i \in \mathbb{N}_n)$. Thus $T\mu \in M_c(L)$. $\square$

The following theorem is the *Lebesgue decomposition theorem*; see [59, Theorem 4.3.2] and [217, Theorem 6.10(a)], for example.

Theorem 4.2.9. *Let K be a non-empty, locally compact space, and suppose that $\mu \in M(K)^+$ and $\nu \in M(K)$. Then there is a unique pair $\{\nu_a, \nu_s\}$ of measures in $M(K)$ with $\nu = \nu_a + \nu_s$, with $\nu_a \ll \mu$, and with $\nu_s \perp \mu$.* $\square$

It is clear that, in the above setting, the maps $\nu \mapsto \nu_a$ and $\nu \mapsto \nu_s$ are Banach-lattice homomorphisms on $M(K)$.

Proposition 4.2.10. *Let K be a non-empty, locally compact space, and suppose that $\mu, \nu \in M(K)^+$ with $\nu \ll \mu$. Then there exists $B \in \mathfrak{B}_K$ with $\nu \sim \mu \mid B$.*

Proof. Take $\mu = \mu_a + \mu_s$ with $\mu_a \ll \nu$ and $\mu_s \perp \nu$, and partition K into two disjoint Borel subsets B and C such that $\mu_s(B) = \nu(C) = 0$. Then

$$(\mu \mid B)(E) = \mu_a(E \cap B) + \mu_s(E \cap B) = \mu_a(E) \quad (E \in \mathfrak{B}_K),$$

and so $\mu \mid B = \mu_a$. Now $\mu_a \sim \nu$ because, for each $A \in \mathfrak{B}_K$ with $\mu_a(A) = 0$, we have $\nu(A) = \nu(A \cap B) = \mu_a(A \cap B) = 0$. Hence $\nu \sim \mu \mid B$, as desired. □

Definition 4.2.11. Let K be a non-empty, locally compact space. For each measure $\mu \in M(K)$, the *disjoint complement* of μ is

$$\mu^{\perp} = \{\nu \in M(K) : \nu \perp \mu\}.$$

It is clear that $\mu^{\perp}$ is a linear subspace of $M(K)$. Further, $\mu \ll \nu$ if and only if $\nu^{\perp} \subset \mu^{\perp}$. The following proposition is easily verified by using elementary vector-lattice exercises.

Proposition 4.2.12. *Let K be a non-empty, locally compact space, and suppose that $\mu, \nu \in M(K)^+$. Then:*

(i) $\mu \perp \nu$ *if and only if* $\mu \wedge \nu = 0$*;*
(ii) $(\mu \vee \nu)^{\perp} = \mu^{\perp} \cap \nu^{\perp} = (\mu + \nu)^{\perp}$*;*
(iii) $\mu^{\perp} \cup \nu^{\perp} \subset (\mu \wedge \nu)^{\perp}$*;*
(iv) $\mu \sim \nu$ *if and only if* $\mu^{\perp} = \nu^{\perp}$. □

Proposition 4.2.13. *Let K be a non-empty, locally compact space, and suppose that F is a complemented face of $P(K)$. Take $\mu \in F$ and $\nu \in F^{\perp}$. Then $\mu \wedge \nu = 0$.*

Proof. Set $\lambda = \mu \wedge \nu$. Clearly $\lambda \leq \nu$, and $\lambda \neq \mu$ because $\mu \not\leq \nu$. Assume towards a contradiction that $\lambda \neq 0$. Then

$$\mu = \|\lambda\| \left(\frac{\lambda}{\|\lambda\|} \right) + \|\mu - \lambda\| \left(\frac{\mu - \lambda}{\|\mu - \lambda\|} \right),$$

and $\|\lambda\| + \|\mu - \lambda\| = 1$ because $\mu - \lambda \geq 0$ and $\|\cdot\|$ is additive on $M(K)^+$. Thus $\lambda / \|\lambda\| \in F$. Similarly, $\lambda / \|\lambda\| \in F^{\perp}$, a contradiction because $F \cap F^{\perp} = \emptyset$. Thus $\lambda = 0$. □

Proposition 4.2.14. *Let K be an infinite, locally compact space. Then*

$$M(K) \cong M(K_\infty).$$

Proof. By equation (4.8), it suffices to show that the subspaces of discrete measures and of continuous measures on K and on K_∞, respectively, are isometrically isomorphic to each other. However, $M_d(K) \cong M_d(K_\infty)$ because $|K| = |K_\infty|$, and, since $\mathfrak{B}_K \subset \mathfrak{B}_{K_\infty}$, the map $\mu \mapsto \mu | \mathfrak{B}_K$ determines a linear isometry from $M_c(K_\infty)$ onto $M_c(K)$. □

Example 4.2.15. Let S be a semigroup. In §1.5, we noted that the space βS becomes a right or left topological semigroup with respect to the operations $\Box$ and $\Diamond$, respectively. Thus the products of u and v in βS are $u \Box v$ and $u \Diamond v$.

The Banach space $(\ell^1(S), \|\cdot\|_1)$ is a Banach algebra with respect to the convolution product $\star$ defined by

$$(f \star g)(t) = \sum\{f(r)g(s) : r, s \in S, rs = t\} \quad (t \in S)$$

for $f, g \in \ell^1(S)$, where we take the sum to be 0 when there are no elements $r, s \in S$ with $rs = t$. It is easily checked that $(\ell^1(S), \|\cdot\|_1, \star)$ is a Banach algebra; it is called the *semigroup algebra* on S.

The bidual of the space $(\ell^1(S), \|\cdot\|_1)$ is identified with the space $M(\beta S)$ of measures on βS, and so the Arens products described in §3.1 give the products $\mu \Box \nu$ and $\mu \Diamond \nu$ for $\mu, \nu \in M(\beta S)$. In particular, we can define the products $\delta_u \Box \delta_v$ and $\delta_u \Diamond \delta_v$ of point masses for $u, v \in \beta S$. These products are easily seen to be consistent with those in βS, in the sense that

$$\delta_u \Box \delta_v = \delta_{u \Box v}, \quad \delta_u \Diamond \delta_v = \delta_{u \Diamond v} \quad (u, v \in \beta S).$$

The Banach algebras $(M(\beta S), \Box)$ and $(M(\beta S), \Diamond)$ are studied in the memoir [71]. In particular, it is shown that $\ell^1(S)$ is usually (but not always) strongly Arens irregular. The interplay between properties of the Banach algebras and the combinatorial properties of the semigroup βS is rather subtle. For further results, see [47]. $\Box$

4.3 A Boolean ring

An introduction to the general theory of Boolean rings and algebras was given in §1.7. We shall now discuss a specific Boolean ring B defined for each non-empty, locally compact space K, with the property that $C^b(St(B)) \cong M(K)'$; this Boolean ring will be used to give a new representation of $C_0(K)''$ in §5.4.

Definition 4.3.1. Let (Ω, Σ, μ) be a measure space. The family of subsets S of Ω such that $\mu(S) = 0$ is denoted by $\mathfrak{N}_\mu$. Then $\Sigma_\mu = \Sigma/\mathfrak{N}_\mu$ and $\pi_\mu : \Sigma \to \Sigma_\mu$ is the quotient map.

Clearly $\mathfrak{N}_\mu$ is a σ-complete ideal in the Boolean algebra Σ, and so Σ_μ is a σ-complete Boolean algebra. We regard μ as a measure on Σ_μ, so that

$$\mu(\pi_\mu(A)) = \mu(A) \quad (A \in \Sigma).$$

In particular, let K be a non-empty, locally compact space, and suppose that $\mu \in M(K)^+$. Then $\mathfrak{B}_\mu = \mathfrak{B}_K/\mathfrak{N}_\mu$. For example, with $K = \mathbb{I}$ and $\mu = m$, we obtain the basic example, $\mathfrak{B}_m$. Note that, when regarded as a function on the Boolean algebra $\mathfrak{B}_\mu$, the measure μ is a σ-normal measure in the sense of Definition 1.7.12.

Proposition 4.3.2. *Let (Ω, Σ, μ) be a finite measure space.*

(i) *Each increasing net $\mathscr{C}$ in Σ_μ has a supremum $B \in \Sigma_\mu$, and*

$$\mu(B) = \sup\{\mu(C) : C \in \mathscr{C}\}.$$

(ii) *The Boolean algebra Σ_μ is complete, and so $St(\Sigma_\mu)$ is a Stonean space.*

(iii) *Suppose that Σ_μ is atomless, and take $B \in \Sigma_\mu$ and $\alpha \in [0, \mu(B)]$. Then there exists $C_0 \in \Sigma_\mu$ with $C_0 \leq B$ and $\mu(C_0) = \alpha$.*

Proof. (i) Choose an increasing sequence (B_n) in $\mathscr{C}$ such that

$$\lim_{n\to\infty} \mu(B_n) = \sup\{\mu(B) : B \in \mathscr{C}\} < \infty,$$

and define $B = \bigvee\{B_n : n \in \mathbb{N}\}$, so that $B \in \Sigma_\mu$ and $\lim_{n\to\infty} \mu(B_n) = \mu(B)$.

We first *claim* that $\mu(C - B) = 0 \ (C \in \mathscr{C})$. Indeed, take $C \in \mathscr{C}$, and assume towards a contradiction that there exists $\delta > 0$ such that $\mu(C - B) > \delta$. Then $\mu(C \vee B_n) > \mu(B_n) + \delta \ (n \in \mathbb{N})$. Choose $m \in \mathbb{N}$ with $\mu(B_m) > \mu(B) - \delta/2$. Since $C \vee B_m \subset D$ for some $D \in \mathscr{C}$, there exists $n \in \mathbb{N}$ such that $\mu(B_n) > \mu(C \vee B_m) - \delta/2$. Thus $\mu(B_n) > \mu(B_m) + \delta/2 > \mu(B)$, the required contradiction. The claim holds.

We next *claim* that $B = \bigvee\{C : C \in \mathscr{C}\}$. By the above paragraph, $C \leq B \ (C \in \mathscr{C})$. Now suppose that $D \in \Sigma_\mu$ is such that $C \leq D \ (C \in \mathscr{C})$. Then $B = \bigvee\{B_n : n \in \mathbb{N}\} \leq D$. It follows that $B = \bigvee\{C : C \in \mathscr{C}\}$, as claimed, and so $\mu(B) = \sup\{\mu(C) : C \in \mathscr{C}\}$.

(ii) It is immediate from (i) that Σ_μ is complete. By Corollary 1.7.5, $St(\Sigma_\mu)$ is a Stonean space.

(iii) Let $\mathscr{C}$ be a chain in Σ_μ such that $\mathscr{C}$ is maximal with respect to the properties that $C \leq B$ and that $\mu(C) \leq \alpha$ whenever $C \in \mathscr{C}$. By (i), there exists $C_0 \in \Sigma_\mu$ with

$$\mu(C_0) = \sup\{\mu(C) : C \in \mathscr{C}\}.$$

Clearly $C_0 \leq B$ and $\mu(C_0) \leq \alpha$. Assume that $\mu(C_0) < \alpha$. Since Σ_μ is atomless, it follows from a remark on page 43 that there is an element $D \in \Sigma_\mu$ with $D \leq B \setminus C_0$ such that $0 < \mu(D) < \alpha - \mu(C_0)$. But now $\mathscr{C} \cup \{C_0 \vee D\}$ is a chain with the property that $\mu(C) \leq \alpha \ (C \in \mathscr{C} \vee \{C_0 \vee D\})$, a contradiction of the maximality of $\mathscr{C}$. Hence $\mu(C_0) = \alpha$. □

Corollary 4.3.3. *Let K be a non-empty, locally compact space.*

(i) *Suppose that $\mu \in P(K)$. Then $\mathfrak{B}_\mu$ is atomless if and only if μ is continuous.*

(ii) *Suppose that $\mu \in M_c(K)^+$ and $\mu \neq 0$. Then $\mathfrak{B}_\mu$ is not a separable Boolean algebra.*

Proof. (i) Suppose that μ is not continuous. Then there exists $x \in K$ such that $\mu(\{x\}) > 0$, and then $\pi_\mu(\delta_x)$ is an atom in $\mathfrak{B}_\mu$.

Suppose that μ is continuous. Then it follows easily from Proposition 4.2.7 that $\mathfrak{B}_\mu$ is atomless.

(ii) Since μ is a non-zero, σ-normal measure on $\mathfrak{B}_\mu$, this follows from Proposition 1.7.13. □

Definition 4.3.4. Let (Ω, Σ, μ) be a probability measure space. We set

$$\rho_\mu(B, C) = \mu(B \Delta C) \quad (B, C \in \Sigma_\mu).$$

It is easy to see that ρ_μ is a metric on the Boolean algebra Σ_μ.

Proposition 4.3.5. *Let (Ω, Σ, μ) be a probability measure space. Then the metric space (Σ_μ, ρ_μ) is complete.*

Proof. As in any metric space, it suffices to show that there exists $B \in \Sigma_\mu$ such that $\rho_\mu(B_k, B) \leq 1/2^k$ $(k \in \mathbb{N})$ whenever $(B_n : n \in \mathbb{N})$ is a sequence in Σ_μ with $\rho_\mu(B_n, B_{n+1}) < 1/2^{n+1}$ $(n \in \mathbb{N})$.

Given such a sequence (B_n), note that $\rho_\mu(B_k, B_n) < 1/2^k$ $(n \geq k)$. For each $n \in \mathbb{N}$, set $D_n = B_n \cup \bigcup_{k \in \mathbb{N}} (B_{n+k-1} \Delta B_{n+k})$. Then $D_n = B_n \cup D_{n+1} \supset D_{n+1}$ and also $B_n \Delta D_n \subset \bigcup_{k \in \mathbb{N}} (B_{n+k-1} \Delta B_{n+k})$, so that $\rho_\mu(B_n, D_n) \to 0$ as $n \to \infty$. Set $B = \bigcap_{n \in \mathbb{N}} D_n$. Then $\mu(D_n) \to \mu(B)$ by the countable additivity of the measure μ, and hence $\rho_\mu(D_n, B) \to 0$ as $n \to \infty$. We have

$$\rho_\mu(B_k, B) \leq \rho_\mu(B_k, B_n) + \rho_\mu(B_n, D_n) + \rho_\mu(D_n, B) \quad (k, n \in \mathbb{N}); \tag{4.11}$$

we fix $k \in \mathbb{N}$, and then take limits in (4.11) as $n \to \infty$ to see that $\rho_\mu(B_k, B) \leq 1/2^k$, giving the result. □

Theorem 4.3.6. *Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be two probability measure spaces such that Σ_{μ_1} and Σ_{μ_2} are atomless Boolean algebras and $(\Sigma_{\mu_1}, \rho_{\mu_1})$ and $(\Sigma_{\mu_2}, \rho_{\mu_2})$ are separable metric spaces. Then there is an isomorphism $\theta : \Sigma_{\mu_1} \to \Sigma_{\mu_2}$ such that*

$$\mu_2(\theta(B)) = \mu_1(B) \quad (B \in \Sigma_1).$$

Proof. Let $\{U_n : n \in \mathbb{N}\}$ and $\{V_n : n \in \mathbb{N}\}$ be countable, dense families in $(\Sigma_{\mu_1}, \rho_{\mu_1})$ and $(\Sigma_{\mu_2}, \rho_{\mu_2})$, respectively, where we write Σ_{μ_i} for $\Sigma_i / \mathfrak{N}_{\mu_i}$ for $i = 1, 2$.

We shall first define increasing sequences $(F_n : n \in \mathbb{N})$ and $(G_n : n \in \mathbb{N})$ of finite Boolean subalgebras of Σ_{μ_1} and Σ_{μ_2}, respectively, and an isomorphism

$$\theta : \bigcup_{n=1}^{\infty} F_n \to \bigcup_{n=1}^{\infty} G_n$$

such that $\mu_2(\theta(B)) = \mu_1(B)$ $(B \in \bigcup_{n=1}^{\infty} F_n)$.

We start by setting $F_1 = \{\emptyset, \Omega_1\}$, $G_1 = \{\emptyset, \Omega_2\}$, $\theta(\emptyset) = \emptyset$, and $\theta(\Omega_1) = \Omega_2$.

Now take $n \in \mathbb{N}$, and assume inductively that $F_1, \ldots, F_n$ and $G_1, \ldots, G_n$ have been defined in Σ_{μ_1} and Σ_{μ_2}, respectively, and that θ has been defined on F_n.

Suppose that n is even, and choose $r \in \mathbb{N}$ to be the smallest number such that $U_r \notin F_n$. By Proposition 4.3.2(iii), for each atom $A \in F_n$, there exists $E_A \in \Sigma_{\mu_2}$ such that $E_A \leq \theta(A)$ and $\mu_2(E_A) = \mu_1(A \wedge U_r)$. We set

$$\theta(A \wedge U_r) = E_A\,, \quad \theta(A - U_r) = \theta(A) - E_A\,,$$

for each such atom A, and we define F_{n+1} to be the (finite) Boolean subalgebra of Σ_{μ_1} generated by $F_n \cup \{U_r\}$; we then extend θ to F_{n+1} in the obvious way, and finally set $G_{n+1} = \theta(F_{n+1})$.

Suppose that n is odd, and choose $r \in \mathbb{N}$ to be the smallest number such that $V_r \notin G_n$. In a similar manner, we extend θ^{-1} to the Boolean subalgebra of Σ_{μ_2} generated by $G_n \cup \{V_r\}$. This completes the inductive construction.

We observe that

$$\theta : \left(\bigcup_{n=1}^{\infty} F_n, \rho_\mu \right) \to \left(\bigcup_{n=1}^{\infty} G_n, \rho_\nu \right)$$

is an isometry and that $\bigcup_{n=1}^{\infty} F_n$ and $\bigcup_{n=1}^{\infty} G_n$ are dense in the metric spaces $(\Sigma_{\mu_1}, \rho_{\mu_1})$ and $(\Sigma_{\mu_2}, \rho_{\mu_2})$, respectively. By Proposition 4.3.5, these two metric spaces are complete, and so the map θ can be extended to an isometry, also called θ, from $(\Sigma_{\mu_1}, \rho_{\mu_1})$ onto $(\Sigma_{\mu_2}, \rho_{\mu_2})$. Clearly θ is an isomorphism between Σ_{μ_1} and Σ_{μ_1}. □

The following consequence of the above theorem, which refers to the measure space $(\mathbb{I}, \Sigma_m, m)$, is sometimes called *von Neumann's isomorphism theorem*. However, the result was essentially known in the 1930s (see Kolmogorov [158, §20] and Szpilrajn [233, Theorem I; note the reference to Jaskowski (1932)]), but apparently the first complete, published proof was by Caratheodory [52, Satz 7 (Hauptsatz)]. Several books now have a proof of this result; a short proof is in Birkhoff [36, p. 262, Corollary]; see also Bogachev [39, Theorem 9.3.4], Halmos [132, §41, Theorem C], and Royden [216, Theorem 15.4].

Corollary 4.3.7. *Let (Ω, Σ, μ) be a probability measure space such that Σ_μ is an atomless Boolean algebra and (Σ_μ, ρ_μ) is a separable metric space. Then there is an isomorphism $\theta : \Sigma_\mu \to \Sigma_m$ such that $m(\theta(B)) = \mu(B)$ $(B \in \Sigma_\mu)$.*

Proof. Since m is a continuous measure, it follows from Corollary 4.3.3(i) that the Boolean algebra Σ_m is atomless, and (Σ_m, ρ_m) is a separable metric space. Now the result follows from Theorem 4.3.6. □

Let K be a non-empty, locally compact space, and take $\mu, \nu \in M(K)$. In Definition 4.2.4, we said that $\mu \sim \nu$ if $\mu \ll \nu$ and $\nu \ll \mu$, so that $\sim$ is an equivalence relation on $M(K)$. The equivalence class containing μ is denoted by $[\mu]$. It is now trivial to check that the relation $\leq$ defined on $M(K)/\sim$ by

$$[\mu] \leq [\nu] \quad \text{if and only if} \quad \mu \ll \nu$$

is a well-defined partial order on $M(K)/\sim$.

We wish to show that the partially ordered space $(M(K)/\sim, \leq)$ is a Boolean ring with certain nice properties. In virtue of the fact that $[\mu] = [|\mu|]$, the space $(M(K)/\sim, \leq)$ is isomorphic to $(M(K)^+/\sim, \leq)$, and so we shall simplify notation and restrict the discussion to positive measures; in particular, for $\mu \in M(K)^+$, we restrict $\mu^\perp$ to $M(K)^+$.

Definition 4.3.8. Let K be a non-empty, locally compact space. We define operations $\vee$ and $\wedge$ on $M(K)^+/\sim$ by:

$$[\mu] \vee [\nu] = [\mu \vee \nu], \quad [\mu] \wedge [\nu] = [\mu \wedge \nu] \quad (\mu, \nu \in M(K)^+).$$

We have to show that the above operations are well defined.

Proposition 4.3.9. *Let K be a non-empty, locally compact space. Then*

$$(M(K)^+/\sim, \leq)$$

is a distributive lattice with a minimum element in which $\vee$ and $\wedge$ are the supremum and infimum in the partial order $\leq$. In particular, $\vee$ and $\wedge$ are well defined.

Proof. Let $\mu, \nu \in M(K)^+$, and set $L = M(K)^+/\sim$ and $S = \{[\mu], [\nu]\}$ in L.

We *claim* that $[\mu \vee \nu]$ is the supremum of S. Indeed, $\mu \ll \mu \vee \nu$ and $\nu \ll \mu \vee \nu$, and so $[\mu \vee \nu]$ is an upper bound for S. Now suppose that $\eta \in M(K)^+$ is such that $[\eta]$ is an upper bound for S. Then $\mu \ll \eta$ and $\nu \ll \eta$, and so $\mu \vee \nu \ll \eta$, whence $[\mu \vee \nu] \leq [\eta]$. The claim follows, and hence $[\mu \vee \nu] = [\mu] \vee [\nu]$.

We also *claim* that $[\mu \wedge \nu]$ is the infimum of S. Indeed, $\mu \wedge \nu \ll \mu$ and $\mu \wedge \nu \ll \nu$, and so $[\mu \wedge \nu]$ is a lower bound for S. Now suppose that $\eta \in M(K)^+$ is such that $[\eta]$ is a lower bound for S, so that $\mu^\perp \subset \eta^\perp$ and $\nu^\perp \subset \eta^\perp$. To show that $[\eta] \leq [\mu \wedge \nu]$, we must show that $(\mu \wedge \nu)^\perp \subset \eta^\perp$. For this, take $\gamma \in (\mu \wedge \nu)^\perp$. Then $\gamma \wedge \mu \wedge \nu = 0$, whence $\gamma \wedge \mu \in \nu^\perp \subset \eta^\perp$, and so $\gamma \wedge \mu \wedge \eta = 0$, i.e., $\gamma \wedge \eta \in \mu^\perp \subset \eta^\perp$. It follows that $\gamma \wedge \eta \wedge \eta = 0 = \gamma \wedge \eta$, and $\gamma \in \eta^\perp$ as desired. The claim follows, and hence $[\mu \wedge \nu] = [\mu] \wedge [\nu]$.

We have shown that L is a lattice. Clearly $[0]$ is the minimum element of L. That L is a distributive lattice follows immediately from the distributivity of the lattice $(M(K)^+, \vee, \wedge)$. □

We remark that an examination of the proof of the preceding proposition shows that an analogous result is valid for any distributive lattice with a minimum element, provided that the relation $a \ll b$ is *defined* by the formula $b^\perp \subset a^\perp$.

Theorem 4.3.10. *Let K be a non-empty, locally compact space, and suppose that $\mu \in M(K)^+$. Then*

$$\{[\nu] : \nu \in M(K)^+, \nu \ll \mu\}$$

is a Boolean algebra in the order $\leq$ inherited from $(M(K)^+/\sim, \leq)$, and it is isomorphic as a Boolean algebra to $\mathfrak{B}_\mu = \mathfrak{B}_K/\mathfrak{N}_\mu$.

Proof. Take $\nu \in M(K)^+$ with $\nu \ll \mu$, and, using the Lebesgue decomposition theorem, Theorem 4.2.9, write $\mu = \mu_a + \mu_s$, where $\mu_a, \mu_s \in M(K)^+$ are such that $\mu_a \ll \nu$ and $\mu_s \perp \nu$. Thus:

$$[\mu_a] \leq [\nu] \leq [\mu]\,; \quad [\mu_s] \leq [\mu]\,; \quad [\mu_s] \wedge [\nu] = [0]\,.$$

We *claim* that $[\nu] \vee [\mu_s] = [\mu]$. Indeed, $[\nu \vee \mu_s] = [\nu + \mu_s]$ by Proposition 4.2.12(ii), and so

$$[\mu] = [\mu_a + \mu_s] \leq [\nu + \mu_s] = [\nu \vee \mu_s] = [\nu] \vee [\mu_s] \leq [\mu]\,,$$

proving the claim.

We have shown that $[\mu_s]$ is the relative complement of $[\nu]$ with respect to $[\mu]$ and that the order interval $[[0],\ [\mu]]$ is a Boolean algebra. Moreover, we observe that $\mu_a \sim \nu$, i.e., $[\mu_a] = [\nu]$, because each is the (unique) relative complement of $[\mu_s]$ with respect to $[\mu]$.

The required Boolean isomorphism is as follows. Take $\nu \in M(K)^+$ with $\nu \ll \mu$. By Proposition 4.2.10, $\nu \sim \mu \mid B$ for some $B \in \mathfrak{B}_K$; the image of ν in $\mathfrak{B}_K/\mathfrak{N}_\mu$ is the equivalence class of B. Note that for $B, C \in \mathfrak{B}_K$, we have $\mu \mid B \sim \mu \mid C$ if and only if $\mu(B \Delta C) = 0$, i.e., if and only if B and C define the same equivalence class in $\mathfrak{B}_\mu$. It is now a simple matter to verify that the map so defined is a bijection which preserves the Boolean operations. □

Theorem 4.3.11. *Let K be a non-empty, locally compact space. Then*

$$(M(K)^+/\sim, \leq)$$

is a Dedekind complete Boolean ring such that, for each $\mu \in M(K)^+$, the order interval $[[0],\ [\mu]]$ is a complete Boolean algebra. Further, the Stone space

$$S_K := St(M(K)^+/\sim, \leq)$$

is an extremely disconnected, locally compact space. For each $\mu \in M(K)^+$, the space $St(\mathfrak{B}_\mu)$ is compact and open in S_K. Further, each compact–open subspace of S_K has the form $St(\mathfrak{B}_\mu)$ for some $\mu \in M(K)^+$, and

$$S_K = \bigcup \{St(\mathfrak{B}_\mu) : \mu \in M(K)^+\}\,.$$

Proof. By Proposition 4.3.9 and Theorem 4.3.10, $(M(K)^+/\sim, \leq)$ is a distributive lattice with a minimum element such that each order interval $[0, \mu]$ is a Boolean algebra, and so it is a Boolean ring.

For each $\mu \in M(K)^+$, the order interval $[[0],\ [\mu]]$ is isomorphic to $\mathfrak{B}_\mu$, which, by Proposition 4.3.2(ii), is a complete Boolean algebra, and so $St(\mathfrak{B}_\mu)$ is a Stonean space. The form of S_K follows from Theorem 1.7.2. Thus $(M(K)^+/\sim, \leq)$ is Dedekind complete and S_K is extremely disconnected. □

4.4 The spaces $L^p(K,\mu)$

We now define the standard spaces $L^\infty(K,\mu)$ and $L^p(K,\mu)$ for $\mu \in M(K)^+$ and p with $1 \leq p < \infty$. In fact, we have already mentioned these spaces when they are defined on a general measure space (Ω,Σ,μ); here we give more details in our special setting.

Let K be a non-empty, locally compact space, and take $\mu \in M(K)^+$. Then two bounded, Borel functions f and g are said to be *equivalent* (with respect to μ) if $\mu(\{x \in K : f(x) \neq g(x)\}) = 0$, or, equivalently, if

$$\int_K |f-g| \,\mathrm{d}\mu = 0;$$

the family of these equivalence classes is the standard Banach space

$$L^\infty(\mu) = L^\infty(K,\mu),$$

with the *essential supremum norm*, $\|\cdot\|_\infty$, so that

$$\|f\|_\infty = \inf\{\alpha > 0 : \mu(\{x \in K : |f(x)| > \alpha\}) = 0\}.$$

The equivalence class containing an element f of $B^b(K)$ is sometimes denoted by $[f]$. The collection of (equivalence classes of) real-valued functions in $L^\infty(\mu)$ is denoted by $L^\infty_{\mathbb{R}}(\mu)$, and the positive functions form the space $L^\infty(\mu)^+$.

We note that $\mathrm{lin}\{[\chi_B] : B \in \mathfrak{B}_K\}$ is a dense linear subspace of $L^\infty(\mu)$.

We remark that every equivalence class in $L^\infty(K,\mu)$ contains a representative in the second Baire class, $B_2(K)$, that was defined in §3.3. This is a classical fact for real functions on an interval in $\mathbb{R}$; see [39, Example 2.12.15] or [116, Theorem 4b, p. 194], for example. The argument in the case of a general locally compact space K and $\mu \in M(K)^+$ follows a parallel route based on Lusin's theorem, Theorem 4.1.7(ii).

Proposition 4.4.1. *Let K be an infinite, locally compact space, and suppose that $\mu \in M(K)^+$ with* $\mathrm{supp}\,\mu = K$. *Then ℓ^∞ is isometrically isomorphic to a 1-complemented subspace of $L^\infty(K,\mu)$.*

Proof. Let (U_n) be a sequence of pairwise-disjoint, non-empty, open subsets of K, so that $\mu(U_n) > 0$ $(n \in \mathbb{N})$. The map

$$(\alpha_n) \mapsto \sum_{n=1}^\infty \alpha_n \chi_{U_n}, \quad \ell^\infty \to L^\infty(K,\mu),$$

is an isometric embedding, with range E, say. The map

$$P : f \mapsto \sum_{n=1}^\infty \frac{1}{\mu(U_n)} \left(\int_{U_n} f \,\mathrm{d}\mu \right) \chi_{U_n}, \quad L^\infty(K,\mu) \to \ell^\infty \cong E,$$

is a bounded projection onto E with $\|P\| = 1$, and so E is a 1-complemented subspace of $L^\infty(K,\mu)$. □

In the following, we shall write $L^\infty(G)$ for $L^\infty(G,m_G)$ when G is a locally compact group G.

Theorem 4.4.2. *Let G be a non-discrete, locally compact group. Then $C^b(G)$ is not complemented in $L^\infty(G)$, and so $C^b(G)$ is not injective.*

Proof. Assume towards a contradiction that there is a bounded projection Q of $L^\infty(G)$ onto the closed subspace $C^b(G)$.

It is standard that there is a compact, symmetric neighbourhood U of e_G such that $G_0 := \bigcup\{U^n : n \in \mathbb{N}\}$ is an infinite, clopen subgroup of G. By replacing G by G_0 and Q by $R \circ (Q \mid L^\infty(G_0))$, where R denotes the restriction map from $C^b(G)$ onto $C^b(G_0)$, we may suppose that G is σ-compact.

By [137, Theorem (8.7)], for each countable family $\{U_n : n \in \mathbb{N}\}$ in $\mathscr{N}_{e_G}$, there is a compact, normal subgroup N of G such that $N \subset \bigcap\{U_n : n \in \mathbb{N}\}$ and the quotient group $H := G/N$ is metrizable; take $\eta : G \to H$ to be the quotient map. Since G is not discrete, we have $m_G(\{e_G\}) = 0$, and so we may suppose that $m_G(N) = 0$; this implies that N is not open in G, and so H is not discrete. Hence there is a sequence (x_n) of distinct points in H with $\lim_{n\to\infty} x_n = e_H$.

For $f \in C^b(G)$, define

$$(Pf)(x) = \int_N f(x\zeta)\,\mathrm{d}m_N(\zeta) \quad (x \in H),$$

so that $Pf \in C^b(H)$ and the map $P : C^b(G) \to C^b(H)$ is a continuous linear surjection. The map

$$R : f \mapsto (f(x_n) - f(e_H)), \quad C^b(H) \to c_0,$$

is also a continuous linear surjection. As before, there exists a sequence (f_n) in $C(H,\mathbb{I})$ with $f_n(x_n) = 1$ $(n \in \mathbb{N})$ and such that $\operatorname{supp} f_m \cap \operatorname{supp} f_n = \emptyset$ when $m,n \in \mathbb{N}$ with $m \neq n$. The map

$$T : \alpha = (\alpha_n) \mapsto \sum_{n=1}^{\infty} \alpha_n (f_n \circ \eta), \quad \ell^\infty \to L^\infty(G),$$

is an isometric embedding, and $T(c_0) \subset C^b(G)$. Thus $S := R \circ P \circ Q \circ T : \ell^\infty \to c_0$ is a bounded operator with $S \mid c_0 = I_{c_0}$. But Phillips' theorem, Theorem 2.4.11, shows that there is no such projection S.

Thus we have a contradiction, and so $C^b(G)$ is not complemented in $L^\infty(G)$. □

For a result related to the above, see [167, Theorem 4].

In fact, it is proved in [198, Theorem 8.9] that, for each infinite, compact group G, the space $C(G)$ is isomorphic to $C(\mathbb{Z}_2^\kappa)$, where $\kappa = w(G)$, so this gives another route to the fact that $C(G)$ is not injective for each infinite, compact group G: as we remarked on page 79, $C(\mathbb{Z}_2^\kappa)$ is not injective. In contrast, there are many compact, non-metrizable spaces K such that $C(K)$ is not isomorphic to a space of the form $C(\mathbb{Z}_2^\kappa)$; such a K can be any infinite Stonean space, or any non-metrizable scattered space, or any space not satisfying CCC [198, Theorem 8.13].

Corollary 4.4.3. *Let G be an infinite, locally compact group. Then $C_0(G)$ is not injective.*

Proof. This follows from Theorem 4.4.2 when G is compact and from Theorem 2.4.12 when G is not pseudo-compact. However a locally compact group that is pseudo-compact as a topological space is already compact. Indeed, take G to be a locally compact, non-compact group, and let K be a compact, symmetric neighbourhood of e_G. Then $K^2 \neq G$: take $x \in G \setminus K^2$. Then $xK \cap K = \emptyset$. Continuing, we find infinitely many, pairwise-disjoint sets x_nK, where $x_n \in G$ $(n \in \mathbb{N})$. For each $n \in \mathbb{N}$, there exists a function $f_n \in C(G, \mathbb{I})$ with $f_n(x_n) = 1$ and $\operatorname{supp} f_n \subset x_nK$, and then $\sum_{n=1}^{\infty} nf_n$ is an unbounded, continuous function on G, and so G is not pseudo-compact. □

Corollary 4.4.4. *Let G be a locally compact group that is extremely disconnected as a topological space. Then G is discrete.*

Proof. By Proposition 1.5.9(ii), βG is Stonean, and so, by Theorem 2.5.11, the space $C^b(G) = C(\beta G)$ is 1-injective. By Theorem 4.4.2, G is discrete. □

In fact, every locally compact group that is an F-space is discrete; for this, see [60, §2.12].

It is clear that each space $L^\infty(K,\mu)$, for a non-empty, locally compact space K and $\mu \in P(K)$, is a commutative, unital C^*-algebra with respect to the pointwise product and conjugation as involution.

Definition 4.4.5. Let K be a non-empty, locally compact space, and suppose that $\mu \in P(K)$. Then the character space of the C^*-algebra $L^\infty(K,\mu)$ is denoted by Φ_μ, and the Gel'fand transform is $\mathscr{G}_\mu : L^\infty(K,\mu) \to C(\Phi_\mu)$.

Thus Φ_μ is a non-empty, compact space and $\mathscr{G}_\mu$ is a unital C^*-isomorphism and a Banach-lattice isometry. It follows that $(C(\Phi_\mu), \leq)$ is a Dedekind complete Banach lattice, and so, by Theorem 2.3.3, Φ_μ is a Stonean space.

Theorem 4.4.6. *Let K be a non-empty, locally compact space, and suppose that $\mu \in P(K)$. Then $L^\infty(K,\mu)$ is a 1-injective space.*

Proof. We know that $L^\infty(K,\mu) \cong C(\Phi_\mu)$ and that Φ_μ is a Stonean space. By Theorem 2.5.11, $C(\Phi_\mu)$ is 1-injective. □

The following is a famous isomorphism theorem of Pełczyński [196].

Theorem 4.4.7. *The spaces ℓ^∞ and $L^\infty(\mathbb{I})$ are isomorphic, so that $\ell^\infty \sim L^\infty(\mathbb{I})$.*

Proof. Set $E = L^\infty(\mathbb{I})$ and $F = \ell^\infty$. By Proposition 2.2.6, $E \sim E \times E$ and $F \sim F \times F$. By Theorem 4.4.6, both E and F are injective spaces. Since E is the dual of $L^1(\mathbb{I})$, it follows from Proposition 2.2.17(iii), there is a linear isometry from E onto a closed subspace of F; by Proposition 4.4.1, there is a linear isometry from F onto a closed subspace of E. It now follows from Proposition 2.5.4 that $E \sim F$. $\square$

The exact Banach–Mazur distance between ℓ^∞ and $L^\infty(\mathbb{I})$ seems to be unknown.

Again let K be a non-empty, locally compact space, and take $\mu \in M(K)^+$. For each p with $1 \leq p < \infty$, we define

$$L^p(K,\mu) = L^p(\mu) = \left\{ f \in \mathbb{C}^K : f \text{ measurable}, \int_K |f|^p \,\mathrm{d}\mu < \infty \right\}$$

and

$$\|f\|_p = \left(\int_K |f|^p \,\mathrm{d}\mu \right)^{1/p} \quad (f \in L^p(\mu)).$$

As usual, we identify equivalent functions f and g, that is, those with $\|f-g\|_p = 0$. Then $(L^p(\mu), \|\cdot\|_p)$ is a Banach space. In particular, with $K = \mathbb{I}$ and $\mu = m$, we obtain the standard Banach spaces $L^p(\mathbb{I})$ of page 5, where we recall that every Lebesgue measurable function on $\mathbb{I}$ is equivalent to a Borel measurable function.

The real-valued and positive functions in $L^p(\mu)$ are denoted by $L^p_{\mathbb{R}}(\mu)$ and $L^p(\mu)^+$, respectively. Again $L^p(\mu)$ is a Dedekind complete Banach lattice: for an explicit proof, see [39, Corollary 4.7.2] or [180, Example 23.3(iv), p. 126], where these spaces are, in fact, shown to be *super-Dedekind complete*, which means that each subset D of these spaces that is bounded above has a supremum which is, moreover, the supremum of some countable subset of D.

We note that $C_0(K)$ and $\mathrm{lin}\{[\chi_B] : B \in \mathfrak{B}_K\}$ are dense linear subspaces of $L^p(\mu)$ for each p with $1 \leq p < \infty$.

Proposition 4.4.8. *Let K be a non-empty, compact, metrizable space, and suppose that $\mu \in M(K)^+$ and $1 \leq p < \infty$. Then $(L^p(K,\mu), \|\cdot\|_p)$ is separable.*

Proof. By Theorem 2.1.7(i), $(C(K), |\cdot|_K)$ is separable, and so this follows because $C(K)$ is dense in $L^p(K,\mu)$. $\square$

The following theorem is the *Radon–Nikodým theorem*; see [39, Theorem 3.2.2], [59, Theorem 4.2.4] and [217, Theorem 6.10(b)], for example.

Theorem 4.4.9. *Let K be a non-empty, locally compact space, and suppose that $\mu \in M(K)^+$ and $\nu \in M(K)$ with $\nu \ll \mu$. Then there is a unique function $h \in L^1(\mu)$ such that*

$$\nu(B) = \int_B h \,\mathrm{d}\mu\,, \quad |\nu|(B) = \int_B |h| \,\mathrm{d}\mu \quad (B \in \mathfrak{B}_K).$$

Further, $\|h\|_1 = \|\nu\|$. In particular, there is a measurable function h on K with $|h(x)| = 1$ $(x \in K)$ and such that $\mathrm{d}\mu = h\,\mathrm{d}|\mu|$. $\square$

Thus, when $\mu, \nu \in M(K)^+$ with $\nu \ll \mu$, we may regard $L^1(\nu)$ as a closed linear subspace of $L^1(\mu)$. Further, we may identify $L^1(\mu)$ with the closed subspace

$$\{\nu \in M(K) : \nu \ll \mu\}$$

of measures in $M(K)$ that are absolutely continuous with respect to μ, so that $L^1(\mu)$ is a lattice ideal in $M(K)$; we have $M(K) = L^1(\mu) \oplus_1 \mu^\perp$, so that $L^1(\mu)$ is 1-complemented in $M(K)$.

The measures on a locally compact group G that are absolutely continuous with respect to left Haar measure m_G are identified with the Banach space

$$L^1(G, m_G),$$

which is regarded as a closed subspace of $M(G)$. This subspace is a closed ideal in the measure algebra $(M(G), \star)$ of G, and it is called the *group algebra* of G; the formula for the product of f and g in $L^1(G, m_G)$ is:

$$(f \star g)(s) = \int_G f(t)g(t^{-1}s)\,\mathrm{d}m_G(t) \quad (s \in G).$$

There is an enormous literature on the group algebra of a locally compact group; it is the central object in the subject 'harmonic analysis'. Again, for example, see the books [68, 137, 194, 195] and the memoir [72].

The following duality theorem is given in [39, §4.4], [59, Proposition 3.5.2], [137, Theorem (12.18)], and [217, Theorem 6.16], for example. For clause (ii), see [138, Theorem (20.20)].

Theorem 4.4.10. (i) *Let (Ω, Σ, μ) be a measure space, and take p with $1 < p < \infty$. Then $(L^p(\Omega, \mu), \|\cdot\|_p)'$ is isometrically isomorphic to $(L^q(\Omega, \mu), \|\cdot\|_q)$, where q is the conjugate index to p. The duality is given by*

$$\langle f, \lambda \rangle = \int_K f\lambda\,\mathrm{d}\mu \quad (f \in L^p(\Omega, \mu),\ \lambda \in L^p(\Omega, \mu)').$$

(ii) *Let (Ω, Σ, μ) be a decomposable measure space. Then $(L^1(\Omega, \mu), \|\cdot\|_1)'$ is isometrically isomorphic to $(L^\infty(\Omega, \mu), \|\cdot\|_\infty)$.* □

Corollary 4.4.11. *Let K be a non-empty, locally compact space, and take $\mu \in P(K)$. Then $L^1(K, \mu)$ is 1-complemented in its bidual*

Proof. We may suppose that $K = \operatorname{supp}\mu$, and so $C_0(K)$ is a closed subspace of $L^\infty(K, \mu)$.

Take $\Lambda \in L^1(K, \mu)''$. Then Λ acts on $L^1(K, \mu)' = L^\infty(K, \mu)$ and hence on $C_0(K)$; we set $R(\Lambda) = \Lambda \mid C_0(K)$, so that R is a bounded projection of $L^1(K, \mu)''$ onto $C_0(K)' = M(K)$ with $\|R\| = 1$. Since $L^1(K, \mu)$ is 1-complemented in $M(K)$, the result follows. □

We now come to a certain uniqueness result for the Banach lattice $L^1(\mathbb{I},m)$. A generalization to the lattices $L^p(\mathbb{I},m)$ for $1 \le p < \infty$ is given in the book [184, Theorem 2.7.3].

Theorem 4.4.12. *Let $(\Omega_1,\Sigma_1,\mu_1)$ and $(\Omega_2,\Sigma_2,\mu_2)$ be probability measure spaces such that Σ_{μ_1} and Σ_{μ_2} are atomless Boolean algebras and the Banach spaces $L^1(\Omega_1,\mu_1)$ and $L^1(\Omega_2,\mu_2)$ are separable. Then there is a Banach-lattice isometry from $L^1(\Omega_1,\mu_1)$ onto $L^1(\Omega_2,\mu_2)$.*

Proof. Since $L^1(\Omega_1,\mu_1)$ and $L^1(\Omega_2,\mu_2)$ are separable Banach spaces, $(\Sigma_{\mu_1},\rho_{\mu_1})$ and $(\Sigma_{\mu_2},\rho_{\mu_2})$ are separable metric spaces. By Theorem 4.3.6. there is an isomorphism $\theta : \Sigma_{\mu_1} \to \Sigma_{\mu_2}$ such that $\mu_2(\theta(B)) = \mu_1(B)$ $(B \in \Sigma_1)$. There is an extension of θ to a linear bijection from $\operatorname{lin}\{\chi_B : B \in \Sigma_1\}$ onto $\operatorname{lin}\{\chi_C : C \in \Sigma_2\}$ with $\theta(\chi_B) = \chi_{\theta(B)}$ $(B \in \Sigma_1)$, and this map is an isometry with respect to the respective norms $\|\cdot\|_1$. Finally, the map θ extends to an isometry from $L^1(\Omega_1,\mu_1)$ onto $L^1(\Omega_2,\mu_2)$. Clearly the final map θ is a lattice isomorphism. □

In fact, let us suppose just that $(\Omega_1,\Sigma_1,\mu_1)$ is a σ-finite measure space. Then, using a remark on page 6, the same conclusion follows.

Corollary 4.4.13. *Let K and L be non-empty, locally compact spaces, and suppose that $\mu \in P_c(K)$ and $\nu \in P_c(L)$ are such that $(L^1(K,\mu),\|\cdot\|_1)$ and $(L^1(L,\nu),\|\cdot\|_1)$ are separable Banach spaces. Then there is a Banach-lattice isometry from $L^1(K,\mu)$ onto $L^1(L,\nu)$.*

Proof. The Boolean algebras $\mathfrak{B}_\mu$ and $\mathfrak{B}_\nu$ are atomless by Corollary 4.3.3(i), and so this is immediate from Theorem 4.4.12. □

Theorem 4.4.14. *Let K be a non-empty, locally compact space, and suppose that $\mu \in P_c(K)$. Then there is an isometric lattice embedding of $L^1(\mathbb{I})$ into $L^1(K,\mu)$. In the case where $(L^1(K,\mu),\|\cdot\|_1)$ is separable, $L^1(K,\mu)$ is Banach-lattice isometric to $L^1(\mathbb{I},m)$.*

Proof. Since the measure μ is continuous, it follows easily from Proposition 4.2.7 that there is a separable, complete, atomless Boolean algebra B contained in $\mathfrak{B}_\mu$. The isomorphism from $\mathfrak{B}_m$ onto B extends to the required isometric lattice embedding. □

Proposition 4.4.15. *Let K be a non-empty, locally compact space.*

(i) *The extreme points of $M(K)_{[1]}$ have the form $\zeta\delta_x$, where $\zeta \in \mathbb{T}$ and $x \in K$, and the extreme points of $P(K)$ have the form δ_x, where $x \in K$.*

(ii) *Take $\mu \in M_c(K)^+$ with $\mu \neq 0$. Then $\operatorname{ex} L^1(\mu)_{[1]} = \emptyset$.*

(iii) *Take $\mu \in M(K)^+$. Then each extreme point of $L^1(\mu)_{[1]}$ has the form $\zeta\delta_x$, where $\zeta \in \mathbb{C}$, $x \in K$, and $|\zeta|\,\mu(\{x\}) = 1$. Further, $\overline{\operatorname{co}}(\operatorname{ex} L^1(\mu)_{[1]}) = L^1(\mu_d)_{[1]}$.*

Proof. (i) Take $\mu \in \operatorname{ex} M(K)_{[1]}$, so that $\mu \neq 0$, and assume towards a contradiction that supp μ is not a singleton. Then there exists $B_0 \in \mathfrak{B}_K$ with $\alpha := |\mu|(B_0) > 0$ and $|\mu|(B_0^c) > 0$, so that $\alpha \in (0,1)$. Define

$$\mu_1(B) = \frac{1}{\alpha}\mu(B \cap B_0), \quad \mu_2(B) = \frac{1}{1-\alpha}\mu(B \cap B_0^c) \quad (B \in \mathfrak{B}_K).$$

Then $\mu_1, \mu_2 \in M(K)_{[1]}$ and $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$, but $\mu_1 \neq \mu$ and $\mu_2 \neq \mu$, a contradiction of the fact that μ is an extreme point of $M(K)_{[1]}$. The result follows.

(ii) Suppose that $f \in L^1(\mu)_{[1]}$ with $\|f\|_1 = 1$. Then there exists $B \in \mathfrak{B}_K$ with

$$0 < \int_B |f| \, d\mu < 1,$$

and now essentially the same argument as above shows that f is a convex combination of two distinct elements of $L^1(\mu)_{[1]}$. Thus $\operatorname{ex} L^1(\mu)_{[1]} = \emptyset$.

(iii) Trivially, the extreme points of $L^1(\mu_d)_{[1]}$ have the form $\zeta\delta_x$, where $\zeta \in \mathbb{C}$, $x \in K$ and $|\zeta| \mu(\{x\}) = 1$. By (ii) and Proposition 2.1.10, $\operatorname{ex} L^1(\mu)_{[1]} = \operatorname{ex} L^1(\mu_d)_{[1]}$, and so the result follows. □

Corollary 4.4.16. *Let K be a non-empty, locally compact space. Then $M_d(K)_{[1]}$ is weak*-dense in $M(K)_{[1]}$.*

Proof. By the Krein–Milman theorem, Theorem 2.6.1, each element of $M(K)_{[1]}$ belongs to the weak*-closure of the convex hull of the set of extreme points of $M(K)_{[1]}$. By the proposition, the extreme points of $M(K)_{[1]}$ belong to $M_d(K)_{[1]}$. □

We saw in Theorem 2.4.15 that c_0 is not isomorphically a dual space: this followed because c_0 is not complemented in its bidual. We now consider the analogous question for the spaces $L^1(K,\mu) = (L^1(K,\mu), \|\cdot\|_1)$, especially in the case where $L^1(K,\mu)$ is separable; by Proposition 4.4.8, the latter case includes that in which K is compact and metrizable. However, we cannot follow the same argument as in the case of c_0 because, by Corollary 4.4.11, $L^1(K,\mu)$ is complemented in its bidual. The fact that the Banach space $L^1(\mathbb{I})$ is not isomorphic to a subspace of a separable dual space was first proved by Gel'fand himself in 1938 [110, p. 265]. The situation for more general spaces $L^1(K,\mu)$ is given below.

Theorem 4.4.17. *Let K be a non-empty, locally compact space, and suppose that $\mu \in P(K)$.*

(i) *The following are equivalent:*

(a) *$L^1(K,\mu)$ is isomorphic to a subspace of a separable dual space;*

(b) *$L^1(K,\mu)$ is isometrically isomorphic to a subspace of a separable dual space;*

(c) *μ is a discrete measure.*

(ii) *The space $L^1(K,\mu)$ is isometrically a dual space if and only if μ is discrete.*

Proof. We may suppose that $L^1(K,\mu)$ is an infinite-dimensional space.

First, suppose that μ is discrete. Then $L^1(K,\mu)$ is isometrically isomorphic to a Banach space of the form

$$\left\{\alpha=(\alpha_n):\|\alpha\|=\sum_{n=1}^{\infty}|\alpha_n|\,\omega_n<\infty\right\}$$

for a sequence (ω_n) in $\mathbb{R}^+\setminus\{0\}$ such that $\sum_{n=1}^{\infty}\omega_n=1$. This space is the dual of the Banach space

$$\{(\beta_n):|\beta_n|/\omega_n\to 0\ \text{ as }\ n\to\infty\},$$

taken with the norm $\|(\beta_n)\|=\sup\{|\beta_n|/\omega_n:n\in\mathbb{N}\}$, and so $L^1(K,\mu)$ is isometrically a dual space.

(i) It is sufficient to show that (a) $\Rightarrow$ (c).

Take a Banach space F with $L^1(K,\mu)\sim F$, where F is a closed subspace of a separable dual space E'. Since E' is separable, E is separable by Proposition 2.1.6. By Corollary 2.6.17, E' has the Krein–Milman property, and so F and $L^1(K,\mu)$ have the Krein–Milman property. Take $\mu_c\in M_c(K)$ and $\mu_d\in M_d(K)$ with $\mu=\mu_c+\mu_d$. Then $L^1(\mu_c)_{[1]}$ is closed, bounded, and convex in $L^1(K,\mu)$, and so, by Proposition 4.4.15(ii), $\mu_c=0$. Hence, $\mu=\mu_d$ is discrete.

(ii) Since $\mu(K)=1$, the set $S:=\{x\in K:\mu(\{x\})>0\}$ is countable. Let T be a countable, dense subset of $\mathbb{T}$. Then, with the identification of Proposition 4.4.15(iii), $\{\zeta\delta_x/\mu(\{x\}):\zeta\in T,x\in S\}$ is a countable, dense subset of $\operatorname{ex}L^1(K,\mu)_{[1]}$, and so $\operatorname{ex}L^1(K,\mu)_{[1]}$ is separable.

Now suppose that $L^1(K,\mu)$ is isometrically a dual space. By Theorem 4.1.10, the space $L^1(K,\mu)$ is separable, and so μ is discrete by (i), (b) $\Rightarrow$ (c). □

Corollary 4.4.18. *Let K be a non-empty, locally compact space, and suppose that $\mu\in M_c(K)^+$ and $L^1(K,\mu)$ is separable. Then there is no embedding of $L^1(K,\mu)$ into a space $\ell^1(D)$ for an index set D.*

Proof. Assume to the contrary that there is an embedding of $L^1(K,\mu)$ into a space $\ell^1(D)$. Since $L^1(K,\mu)$ is separable, there is a countable subset D_0 of D such that $L^1(K,\mu)$ embeds into $\ell^1(D_0)$, a separable dual space. This is a contradiction of Theorem 4.4.17(i), (a) $\Rightarrow$ (c). □

The above theorem gives *Gel'fand's theorem*, which we state explicitly.

Theorem 4.4.19. *The Banach space $L^1(\mathbb{I})$ is not isomorphic to a subspace of a separable dual space. In particular, $L^1(\mathbb{I})$ is not isomorphically a dual space.* □

There is a different, self-contained proof of the above theorem, along with some informative remarks, in [3, Theorem 6.3.7].

An alternative proof that the space $L^1(K,\mu)$ of Corollary 4.4.18 does not embed in ℓ^1 is mentioned after Corollary 4.5.8, below.

Let K be a non-empty, locally compact space. Using more sophisticated techniques than the above, Pełczyński showed in [197] that, for a σ-finite positive measure μ, the space $L^1(K,\mu)$ is isomorphically a dual space if and only if μ is discrete.

See also [168] and [211]. A different proof, for the case of *finite* measures, is given in [85, p. 83]. For positive measures μ on K that are not σ-finite, it seems to be unknown which $L^1(K,\mu)$ spaces are isomorphically dual spaces. In the isometric theory, an early result of this type is given in [94, Exercise 4, p. 458]. Let (Ω,μ) be a measure space, where μ is a σ-finite positive measure. Then $L^1(\Omega,\mu)$ is isometrically a dual space if and only if Ω is a countable union $\Omega = \bigcup \Omega_i$, where each Ω_i is a measurable subset of Ω with $\mu(\Omega_i) < \infty$ and such that, for each measurable subset A of each Ω_i, we have either $\mu(A) = 0$ or $\mu(A) = \mu(\Omega_i)$. Suppose that, in fact, $\mu(\{x\}) = 1$ for each $x \in \Omega$. Then it follows that $L^1(\Omega,\mu) \cong \ell^1$.

We conclude this section with two well-known results on weak compactness in L^1-spaces that we shall use. The first proposition is a result on equi-continuity.

Proposition 4.4.20. *Let K be a non-empty, compact space, and take $\nu \in M(K)^+$. Suppose that (μ_n) is a sequence in $L^1(K,\nu)$ that converges weakly. Then, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $|\mu_n|(B) \leq \varepsilon$ $(n \in \mathbb{N})$ whenever $B \in \mathfrak{B}_K$ with $\nu(B) \leq \delta$.*

Proof. We may suppose that $\nu \in P(K)$. By Proposition 4.3.5, the metric space $(\mathfrak{B}_\nu, \rho_\nu)$ is complete.

First, suppose that (μ_n) converges weakly to 0. Fix $\varepsilon > 0$, and, for $n \in \mathbb{N}$, set

$$G_n = \{B \in \mathfrak{B}_\nu : |\mu_m(B)| \leq \varepsilon \quad (m \geq n)\} .$$

Then each set G_n is closed in the space $(\mathfrak{B}_\nu, \rho_\nu)$, and $\bigcup\{G_n : n \in \mathbb{N}\} = \mathfrak{B}_\nu$ because $\lim_{n\to\infty} \mu_n(B) = 0$ for each $B \in \mathfrak{B}_K$. By Baire's theorem, Theorem 1.4.11, there exist $n_0 \in \mathbb{N}$, $B_0 \in \mathfrak{B}_K$, and $\delta_0 > 0$ such that $|\mu_n(B)| < \varepsilon$ whenever $n \geq n_0$ and $B \in \mathfrak{B}_K$ with $\rho_\nu(B,B_0) < \delta_0$.

Suppose that $B \in \mathfrak{B}_K$ with $\nu(B) < \delta_0$. Then $\rho_\nu(B_0 \cup B, B_0) = \nu(B \setminus B_0) < \delta_0$ and $\rho_\nu(B_0 \setminus B, B_0) = \nu(B_0 \cap B) < \delta_0$, and so

$$|\mu_n(B)| \leq |\mu_n(B_0 \cup B)| + |\mu_n(B_0 \setminus B)| < 2\varepsilon \quad (n \geq n_0).$$

By inequality (4.10), $|\mu_n|(B) \leq 8\varepsilon$ $(n \geq n_0)$. By reducing δ_0, if necessary, we may suppose that the same inequality holds for each $n \in \mathbb{N}_{n_0}$, and hence for all $n \in \mathbb{N}$. The result now follows in this special case.

Now suppose that (μ_n) converges weakly to some limit in $M(K)$. We *claim* that, for each $\varepsilon > 0$, there exist $\delta_0 > 0$ and $n_0 \in \mathbb{N}$ such that $|\mu_m - \mu_n|(B) \leq \varepsilon/2$ whenever $m,n \geq n_0$ and $B \in \mathfrak{B}_K$ with $\nu(B) \leq \delta_0$. Assume that this is not the case. Then there exist $\varepsilon > 0$, strictly increasing sequences (m_k) and (n_k) in $\mathbb{N}$, and sets B_k in $\mathfrak{B}_K$ such that $\nu(B_k) \leq 1/k$ and $\left|\mu_{m_k} - \mu_{n_k}\right|(B_k) \geq \varepsilon$ for each $k \in \mathbb{N}$. Since

$$\lim_{k\to\infty} (\mu_{m_k} - \mu_{n_k})(B) = 0 \quad (B \in \mathfrak{B}_K),$$

this contradicts the result in the special case. Thus the claim holds.

Finally, choose $\delta \in (0,\delta_0)$ such that $|\mu_n|(B) < \varepsilon/2$ whenever $n \in \mathbb{N}_{n_0}$ and $B \in \mathfrak{B}_K$ with $\nu(B) \leq \delta$. Then the required conclusion follows. □

Theorem 4.4.21. *Let K be a non-empty, compact space, and take $\nu \in M(K)^+$. Suppose that S is a subset of $L^1(K,\nu)$. Then S is relatively weakly compact if and only if:*

(i) *S is norm-bounded;*

(ii) *for each $\varepsilon > 0$, there exists $\delta > 0$ such that $|\mu(B)| < \varepsilon$ $(\mu \in S)$ whenever $B \in \mathfrak{B}_K$ with $\nu(B) \le \delta$.*

Proof. Suppose that S is relatively weakly compact. Then S is weakly bounded, and hence norm-bounded by Corollary 2.2.2, so that (i) holds. Assume towards a contradiction that (ii) fails. Then there exist $\varepsilon > 0$, a sequence (μ_n) in S, and a sequence (B_n) in $\mathfrak{B}_K$ with $\nu(B_n) \le 1/n$ and $|\mu_n(B_n)| > \varepsilon$ for all $n \in \mathbb{N}$. By the Eberlein–Šmulian theorem, Theorem 2.1.4(vii), (μ_n) has a weakly convergent subsequence, say (μ_{n_k}). By Proposition 4.4.20, there exists $\delta > 0$ with $\left|\mu_{n_k}\right|(B) < \varepsilon/2$ $(k \in \mathbb{N})$ whenever $B \in \mathfrak{B}_K$ with $\nu(B) \le \delta$. Take $k \in \mathbb{N}$ with $1/n_k < \delta$. Then

$$\varepsilon \le \left|\mu_{n_k}(B_{n_k})\right| \le \left|\mu_{n_k}\right|(B_{n_k}) \le \frac{\varepsilon}{2},$$

a contradiction. Thus (ii) holds.

Conversely, suppose that S satisfies clauses (i) and (ii). We regard $E := L^1(K,\nu)$ and S as subsets of E''. Then S is norm-bounded in E'', and so has a weak*-limit point, say M, in E''. Define

$$\lambda(B) = \langle \chi_B, \mathrm{M} \rangle \quad (B \in \mathfrak{B}_K).$$

Take $\varepsilon > 0$, and choose $\delta = \delta(\varepsilon) > 0$ as specified in (ii). Now take $\eta > 0$. For each $B \in \mathfrak{B}_K$ with $\nu(B) \le \delta$, we have $\chi_B \in E'$, and so there exists $\mu \in S$ with

$$|\langle \chi_B, \mathrm{M} \rangle - \mu(B)| < \eta,$$

and then $|\lambda(B)| \le \varepsilon + \eta$. This holds for each $\eta > 0$, and so $|\lambda(B)| \le \varepsilon$.

Suppose that (B_n) is a sequence in $\mathfrak{B}_K$ with $\nu(B_n) \searrow 0$. Then $|\lambda(B_n)| \searrow 0$, and so λ is countably additive on $\mathfrak{B}_K$, and hence $\lambda \in M(K)$. Also $\lambda \ll \nu$, and so, by the Radon–Nikodým theorem, Theorem 4.4.9, $\lambda \in E$. It follows that M is a weak-limit point of S in E, and hence that S is relatively weakly compact. □

4.5 The space $C(K)$ as a Grothendieck space

We now consider when a space $C(K)$ for K compact is a Grothendieck space. Of course we have characterized such spaces in the (unproved) Proposition 2.4.7. We shall show in Corollary 4.5.10 that $C(K)$ is a Grothendieck space whenever it is an injective space.

First note that $C(K)$ is certainly not a Grothendieck space whenever K contains a convergent sequence (x_n) of distinct points, say with limit $x \in K$. Indeed, the sequence $(\delta_{x_n} - \delta_x)$ in $M(K)$ converges weak* to 0, but it does not converge weakly to 0, as can be seen by considering the linear functional $\mu \mapsto \sum_{n=1}^{\infty} \mu(\{x_n\})$ on $M(K)$.

We shall also use the following result of Grothendieck from [124] about relative weak compactness in the Banach space $M(K)$.

Theorem 4.5.1. *Let K be a non-empty, compact space, and take S to be a norm-bounded subset of $M(K)$. Then the following conditions are equivalent:*

(a) *S is relatively weakly compact;*

(b) *for each sequence (μ_n) in S, necessarily $\lim_{n\to\infty} \mu_n(U_n) = 0$ for each sequence (U_n) of pairwise-disjoint, open sets in K.*

An early proof of this theorem is contained in Bade's notes [24, §9]; see also [3, §5.3], [94, Theorem IV.9.1], and [184, Theorem 2.5.5], for example.

We shall first prove two lemmas, in which we suppose that the set S is a norm-bounded subset of $M(K)$ that satisfies clause (b) of Theorem 4.5.1.

Lemma 4.5.2. *Let (μ_n) be a sequence in S. Then $\lim_{n\to\infty} |\mu_n|(U_n) = 0$ for each sequence (U_n) of pairwise-disjoint, open sets in K.*

Proof. For $n \in \mathbb{N}$, take ν_n to be either $\Re\mu_n$ or $\Im\mu_n$. Then $\lim_{n\to\infty} \nu_n(U_n) = 0$ for each sequence (U_n) of pairwise-disjoint, open sets in K.

Assume to the contrary that there is a sequence (U_n) of pairwise-disjoint, open sets in K such that $(|\nu_n|(U_n))$ does not converge to 0. Set $\nu_n = \nu_n^+ - \nu_n^-$ $(n \in \mathbb{N})$; we may suppose that $(\nu_n^+(U_n))$ does not converge to 0, and, by passing to a subsequence, we may suppose that there exists $\delta > 0$ with $\nu_n^+(U_n) > \delta$ $(n \in \mathbb{N})$. By Hahn's decomposition theorem, Theorem 4.1.7(i), for each $n \in \mathbb{N}$, there is a Borel subset B_n of U_n with $\nu_n(B_n) = \nu_n^+(U_n)$, and, by the regularity of ν_n, there is an open set V_n with $B_n \subset V_n \subset U_n$ and $\nu_n(V_n) > \delta$, a contradiction.

Thus $\lim_{n\to\infty} |\nu_n|(U_n) = 0$ for each sequence (U_n) of pairwise-disjoint, open sets, and then the result follows. □

The second lemma states that the subset S of $M(K)$ is *uniformly regular*.

Lemma 4.5.3. *For each compact subset L of K and each $\varepsilon > 0$, there is an open subset U of K with $U \supset L$ such that $|\mu|(U \setminus L) \le \varepsilon$ $(\mu \in S)$.*

Proof. Assume that the conclusion fails. Then there is a compact subset L of K and $\varepsilon > 0$ such that, for each open neighbourhood U of L, there exists $\mu \in S$ with $|\mu|(U \setminus L) > \varepsilon$.

We *claim* that there are a sequence (W_n) of open subsets of K such that the sets $\overline{W_n}$ are contained in $K \setminus L$ and are pairwise disjoint and a sequence (μ_n) in S such that $|\mu_n(W_n)| > \varepsilon/4$ $(n \in \mathbb{N})$.

Indeed, take $V_1 = K$, and choose $\mu_1 \in S$ with $|\mu_1|(V_1 \setminus L) > \varepsilon$. By the regularity of $|\mu_1|$, there is an open set W_1 in K with $\overline{W_1} \subset V_1 \setminus L$ and with $|\mu_1(W_1)| > \varepsilon/4$, where we are using inequality (4.10). Now take $k \in \mathbb{N}$, and assume that $W_1, \dots, W_k$ and $\mu_1, \dots, \mu_k$ have been determined to satisfy the claim for each $n \in \mathbb{N}_k$. Set $V_{k+1} = \bigcup_{j=1}^{k}(K \setminus \overline{W_j})$, and then choose $\mu_{k+1} \in S$ and an open set W_{k+1} such that $\overline{W_{k+1}} \subset V_{k+1} \setminus L$ and $|\mu_{k+1}(W_{k+1})| > \varepsilon/4$. This continues the inductive construction, and hence the claim holds.

However, the claim contradicts clause (b) of Theorem 4.5.1, and so the conclusion holds. □

Proof of Theorem 4.5.1. We first show that clause (b) of Theorem 4.5.1 holds whenever S is relatively weakly compact.

Indeed, take a sequence (μ_n) in S. By the Eberlein–Šmulian theorem, Theorem 2.1.4(vii), we may suppose, by passing to a subsequence, that (μ_n) converges weakly in $M(K)$. Define

$$\nu = \sum_{n=1}^{\infty} \frac{|\mu_n|}{2^n} \in M(K)^+ . \tag{4.12}$$

For each $n \in \mathbb{N}$, we have $\mu_n \ll \nu$, and so, by the Radon–Nikodým theorem, Theorem 4.4.9, we may suppose that $\mu_n \in L^1(K, \nu)$ $(n \in \mathbb{N})$. Clearly the sequence (μ_n) converges weakly in $L^1(K, \nu)$, and so, by Proposition 4.4.20, clause (b) holds.

We now show that clause (b) implies that S is relatively weakly compact.

By the Eberlein–Šmulian theorem, it is sufficient to show that each countable subset of S is relatively weakly compact in $M(K)$; we take such a countable set $T := \{\mu_n : n \in \mathbb{N}\}$, and define ν as in equation (4.12). Clearly, it suffices to show that the set T is relatively weakly compact in $L^1(K, \nu)$; for this, we shall show that T satisfies clauses (i) and (ii) of Theorem 4.4.21.

By hypothesis, S is norm-bounded in $M(K)$, and so T satisfies clause (i) of 4.4.21.

Assume towards a contradiction that T does not satisfy clause (ii). Then, by using the regularity of ν and passing to a subsequence of (μ_n), we may suppose that there are $\varepsilon > 0$ and a sequence (B_n) of sets in $\mathfrak{B}_K$ such that

$$\nu(B_n) \le \frac{1}{n} \quad \text{and} \quad |\mu_n|(B_n) \ge |\mu_n(B_n)| > \varepsilon$$

for all $n \in \mathbb{N}$.

For each $m \in \mathbb{N}$, we have $\lim_{n\to\infty} |\mu_m|(B_n) = 0$, and so, by passing to a further subsequence, we may suppose that

$$|\mu_m|(B_n) < \frac{\varepsilon}{2^{n+2}} \quad (n > m,\ m, n \in \mathbb{N}) .$$

Take $m \in \mathbb{N}$, and set $C_m = B_m \setminus \bigcup\{B_n : n \ge m+1\}$. Then C_m is a Borel subset of B_m such that $|\mu_m|(C_m) > \varepsilon/2$. Further, the sets C_m are pairwise disjoint. By the regularity of the measures μ_m, we can choose compact subsets L_m of C_m such that $|\mu_m|(L_m) > \varepsilon/2$. It follows from Lemma 4.5.3 that there is an open set W_m with $W_m \supset L_m$ such that $|\mu_n|(W_m \setminus L_m) < \varepsilon/2^{m+4}$ $(n \in \mathbb{N})$. We can then choose an open set V_m such that $L_m \subset V_m \subset \overline{V_m} \subset W_m$.

Now take $m,n \in \mathbb{N}$ with $m < n$. Then

$$(\overline{V_m} \cap V_n) \subset (\overline{V_m} \setminus L_m) \cup (V_n \setminus L_n) \subset (W_m \setminus L_m) \cup (W_n \setminus L_n),$$

and so $|\mu_n|(V_n \cap \overline{V_m}) < \varepsilon/2^{m+3}$. For $n \geq 2$, set $G_n = V_n \setminus \overline{V_1 \cup \cdots \cup V_{n-1}}$. Then the sequence $(G_n : n \geq 2)$ consists of pairwise-disjoint, open subsets of K, and $|\mu_n|(G_n) > \varepsilon/2 - \varepsilon/4 = \varepsilon/4$. This is a contradiction of Lemma 4.5.2, and so T satisfies clause (ii) of Theorem 4.4.21. By Theorem 4.4.21, T is relatively weakly compact in $L^1(K,\nu)$, as required. □

Corollary 4.5.4. *Let K be a non-empty, compact space, and take $\nu \in M(K)$. Then the set $\{\mu \in M(K) : |\mu| \leq |\nu|\}$ is weakly compact.*

Proof. This result follows immediately from Theorem 4.5.1. □

We shall use Corollary 4.5.4 to give the following direct, elementary proof that each space $C_0(K)$ is Arens regular; in fact, this result will also follow from the construction of the bidual of $C_0(K)$, to be given in Theorem 5.4.1.

Theorem 4.5.5. *Let K be a non-empty, locally compact space. Then the C^*-algebra $C_0(K)$ is Arens regular, and $(C_0(K)'', \Box)$ is commutative.*

Proof. Take $\mathrm{M} \in C_0(K)'' = M(K)'$ and $\mu \in C_0(K)'_{[1]} = M(K)_{[1]}$, and consider the continuous linear functional

$$\theta : \mathrm{N} \mapsto \langle \mathrm{M} \Box \mathrm{N}, \mu \rangle = \langle \mathrm{M}, \mathrm{N} \cdot \mu \rangle, \quad M(K)' \to \mathbb{C}.$$

We *claim* that θ is weak*-continuous on $M(K)'_{[1]}$. For suppose that $\mathrm{N}_\alpha \to \mathrm{N}_0$ in $(M(K)'_{[1]}, \sigma(M(K)', M(K)))$. Then $(\mathrm{N}_\alpha \cdot \mu)$ is a net in $\{\nu \in M(K) : |\nu| \leq |\mu|\}$; by Corollary 4.5.4, this latter set is weakly compact, and so $(\mathrm{N}_\alpha \cdot \mu)$ has a weakly convergent subnet, say $(\mathrm{N}_{\alpha_\beta} \cdot \mu)$. For each $f \in C_0(K)$, we have

$$\langle f, \mathrm{N}_0 \cdot \mu \rangle = \langle \mathrm{N}_0, \mu \cdot f \rangle = \lim_\alpha \langle \mathrm{N}_\alpha, \mu \cdot f \rangle = \lim_\beta \langle \mathrm{N}_{\alpha_\beta}, \mu \cdot f \rangle = \lim_\beta \langle f, \mathrm{N}_{\alpha_\beta} \cdot \mu \rangle,$$

and hence $\lim_\beta \mathrm{N}_{\alpha_\beta} \cdot \mu = \mathrm{N}_0 \cdot \mu$ in $(M(K), \sigma(M(K), C_0(K)))$. This implies that the net $(\mathrm{N}_\alpha \cdot \mu)$ converges weakly to $\mathrm{N}_0 \cdot \mu$, and so

$$\lim_\alpha \theta(\mathrm{N}_\alpha) = \lim_\alpha \langle \mathrm{M}, \mathrm{N}_\alpha \cdot \mu \rangle = \langle \mathrm{M}, \mathrm{N}_0 \cdot \mu \rangle = \theta(\mathrm{N}_0),$$

giving the claim.

It follows from Theorem 2.1.4(iv), (c) $\Rightarrow$ (a), that there exists $\nu \in M(K)$ such that

$$\theta(\mathrm{N}) = \langle \mathrm{N}, \nu \rangle \quad (\mathrm{N} \in M(K)').$$

For each $f \in C_0(K)$, we have $\langle f, \nu \rangle = \langle \mathrm{M} \cdot f, \mu \rangle = \langle \mathrm{M}, f \cdot \mu \rangle = \langle f, \mu \cdot \mathrm{M} \rangle$, and so $\nu = \mu \cdot \mathrm{M}$. We have shown that

$$\langle \mathrm{M} \,\Box\, \mathrm{N}, \mu \rangle = \theta(N) = \langle \mathrm{N}, \mu \cdot \mathrm{M} \rangle = \langle \mathrm{M} \Diamond \mathrm{N}, \mu \rangle \quad (\mathrm{M}, \mathrm{N} \in C_0(K)'', \mu \in C_0(K)'),$$

and hence $\mathrm{M} \,\Box\, \mathrm{N} = \mathrm{M} \Diamond \mathrm{N}$ $(\mathrm{M}, \mathrm{N} \in C_0(K)'')$. Thus $C_0(K)$ is Arens regular.

Since $C_0(K)$ is commutative, $(C_0(K)'', \Box)$ is commutative. □

The next result is a classic theorem of Grothendieck [124]. Grothendieck's proof utilized a lemma of Phillips [202] on sequential convergence in the space of finitely additive measures on $\mathscr{P}(\mathbb{N})$, as described in [24]; we give a direct and self-contained proof.

Theorem 4.5.6. *Let K be a Stonean space. Then $C(K)$ is a Grothendieck space.*

Proof. Let (μ_n) be a sequence in $C(K)' = M(K)$ that converges weak* to 0; we must show that (μ_n) converges weakly, and, for this, it suffices to show that the set $\{\mu_n : n \in \mathbb{N}\}$ is relatively weakly compact in $M(K)$.

Assume to the contrary that this fails. Then, it follows from Theorem 4.5.1 that, after passing to a subsequence and rescaling, we may suppose that there is a pairwise-disjoint sequence (U_n) of open subsets of K with $|\mu_n(U_n)| > 1$ $(n \in \mathbb{N})$. Since K is Stonean and each μ_n is regular, we may suppose that all the sets U_n are clopen.

We shall define inductively a subsequence (μ_{n_k}) of (μ_n) such that (n_k) is strictly increasing in $\mathbb{N}$ and

$$|\mu_{n_r}(U_{n_s})| < \frac{1}{2^{s+1}} \quad (r, s \in \mathbb{N}, r \neq s). \tag{4.13}$$

First, take $n_1 = 1$. Now suppose that $k \in \mathbb{N}$, and assume that $n_1, \dots, n_k$ have been defined such that (4.13) holds whenever $r, s \in \mathbb{N}_k$ and $r \neq s$. For each $j \in \mathbb{N}_k$, the set

$$\left\{ n \in \mathbb{N} : \left|\mu_{n_j}(U_n)\right| \geq \frac{1}{2^{k+2}} \right\}$$

is finite and $\lim_{n\to\infty} \mu_n(U_{n_j}) = 0$, and so we can choose $n_{k+1} > n_k$ such that $\left|\mu_{n_j}(U_{n_{k+1}})\right| < 1/2^{k+2}$ and $\left|\mu_{n_{k+1}}(U_{n_j})\right| < 1/2^{j+1}$ for $j \in \mathbb{N}_k$. This continues the inductive construction of the sequence (n_k). The sequence satisfies (4.13); set $\nu_k = \mu_{n_k}$ and $V_k = U_{n_k}$ for $k \in \mathbb{N}$.

As in Proposition 1.5.5, there are an index set A such that $|A| = \mathfrak{c}$ and a family $\{S_\alpha : \alpha \in A\}$ of infinite subsets of $\mathbb{N}$ such that $S_\alpha \cap S_\beta$ is finite whenever $\alpha, \beta \in A$ with $\alpha \neq \beta$. For each $\alpha \in A$, set

$$W_\alpha = \overline{\bigcup\{V_k : k \in S_\alpha\}},$$

a clopen subset of K, and set $V = \bigcup\{V_k : k \in \mathbb{N}\}$, an open subset of K. We note that $\{W_\alpha \setminus V : \alpha \in A\}$ is a family of pairwise-disjoint, closed subsets of K. For each $k \in \mathbb{N}$, it is the case that $\nu_k(W_\alpha \setminus V) \neq 0$ for only countably many values of $\alpha \in A$, and so there exists $\alpha \in A$ with $\nu_k(W_\alpha \setminus V) = 0 \ (k \in \mathbb{N})$. Thus, for each $k \in S_\alpha$, we have

$$|\langle \chi_{W_\alpha}, \nu_k \rangle| = |\nu_k(W_\alpha \cap V)| \geq |\nu_k(V_k)| - \sum \left\{ \left|\nu_k(V_j)\right| : j \in \mathbb{N}, \, j \neq k \right\} > 1/2\,,$$

using (4.13), a contradiction of the fact that (ν_k) converges weak* to 0.

The result follows. □

Definition 4.5.7. A Banach space E has the *Schur property* if every weakly convergent sequence in E is norm-convergent.

Corollary 4.5.8. *Let S be a non-empty set. Then $\ell^\infty(S)$ is a Grothendieck space. Further, suppose that (μ_n) is sequence in $M(\beta S)$ that is weak*-convergent to 0. Then*

$$\lim_{n\to\infty} \|\mu_n \mid S\| = 0\,,$$

and $\ell^1(S)$ has the Schur property.

Proof. Since $\ell^\infty(S) \cong C(\beta S)$ and βS is a Stonean space, certainly $\ell^\infty(S)$ is a Grothendieck space by Theorem 4.5.6.

Suppose that (μ_n) in $M(\beta S)$ is weak*-convergent to 0, and assume towards a contradiction that it is not true that $\lim_{n\to\infty} \|\mu_n \mid S\| = 0$. By passing to a subsequence and rescaling, we may suppose that $\|\nu_n\| > 1 \ (n \in \mathbb{N})$, where $\nu_n = \mu_n \mid S$. Essentially as in the above proof, there is a sequence (F_n) of pairwise-disjoint, finite subsets of S such that $|\mu_n(F_n)| = |\nu_n(F_n)| > 1 \ (n \in \mathbb{N})$. By Theorem 4.5.1, the sequence (μ_n) is not relatively weakly compact, and this contradicts Theorem 4.5.6.

In the case where (μ_n) is weakly convergent to 0 in $\ell^1(S)$, it follows that (μ_n), regarded as a sequence in $M(\beta S)$, is weak*-convergent to 0, and so (μ_n) is norm-convergent to 0 in $\ell^1(S)$. □

The fact that ℓ^1 has the Schur property goes back to Schur in 1921 and is included in Banach's book [30, Table (property 17), p. 245; also, p. 239]; for a modern discussion, see [2, Theorem 2.3.6 and p. 102].

It is easily seen that $L^1(\mathbb{I})$ does not have the Schur property, and hence also that the spaces $L^1(K,\mu)$ for K locally compact and $\mu \in P_c(K)$ do not have the Schur property. Indeed, consider the sequence (s_n) of page 116. This sequence is weakly convergent to 0 in $L^1(\mathbb{I})$. However, (s_n) is certainly not norm-convergent to 0 in $L^1(\mathbb{I})$. Hence $L^1(K,\mu)$ does not embed in ℓ^1.

The above results give a slightly different proof of Phillips' theorem, Theorem 2.4.11. Indeed, assume towards a contradiction that $P : \ell^\infty \to c_0$ is a bounded projection, so that $P' : c_0' \to M(\beta\mathbb{N})$ is a bounded operator. Regard δ_n as a continuous linear functional on c_0 for $n \in \mathbb{N}$. Then

$$\langle f, P'(\delta_n)\rangle = \langle Pf, \delta_n\rangle \to 0 \quad \text{as} \quad n \to \infty \quad (f \in \ell^\infty)\,,$$

and so $P'(\delta_n) \to 0$ weak* in $M(\beta\mathbb{N})$. By Corollary 4.5.8, $|P'(\delta_n)(\{n\})| \to 0$ as $n \to \infty$. But $P'(\delta_n)(\{n\}) = 1$ $(n \in \mathbb{N})$, a contradiction.

The following corollary of Theorem 4.5.6 was noted by Seever in [224]; see also [184, Corollary 2.5.17].

Corollary 4.5.9. *Let K be a compact F-space. Then $C(K)$ is a Grothendieck space.*

Proof. Let (μ_n) be a sequence in $M(K) = C(K)'$ that converges weak* to 0, and define $\mu = \sum_{n=1}^{\infty} \mu_n/2^n \in M(K)$. Set $L = \operatorname{supp} \mu$. By Proposition 4.1.6, L is a Stonean space. Then, by Theorem 4.5.6, $(\mu_n \mid L)$ converges weak* to 0 in $M(L)$, and so it converges weakly to 0 in $M(L)$, i.e., (μ_n) converges weakly to 0 in $M(K)$. Hence $C(K)$ is a Grothendieck space. □

Corollary 4.5.10. *Each injective space is a Grothendieck space.*

Proof. Let E be a Banach space. By Proposition 2.2.14(i), there is a set S and an isometric embedding of E onto a subspace, say F, of $\ell^\infty(S)$. In the case where E is injective, F is complemented in $\ell^\infty(S)$. Since $\ell^\infty(S)$ is a Grothendieck space and complemented subspaces of Grothendieck spaces are also Grothendieck spaces (see page 73), E is a Grothendieck space. □

We shall see in Example 6.8.17 that there are compact spaces K such that $C(K)$ is a Grothendieck space, but $C(K)$ is not injective. The Baire classes $B_\alpha(\mathbb{I})$ for ordinals α with $1 \leq \alpha \leq \omega_1$ are examples of $C(K)$-spaces that are Grothendieck spaces (see Theorem 3.3.9), but are such that K is not an F-space when $\alpha < \omega_1$ [76].

A beautiful generalization of Theorem 4.5.1 characterizing weak compactness in the dual of a C^*-algebra was given by Pfitzner in [200]. For a shorter proof, see [101]; see also [2]. It follows that each von Neumann algebra is a Grothendieck space; it is proved in [219] that each monotone σ-complete C^*-algebra is a Grothendieck space.

4.6 Singular families of measures

We now introduce singular families and maximal singular families of measures.

Definition 4.6.1. Let K be a non-empty, locally compact space. A family $\mathscr{F}$ of measures in $M(K)^+$ is *singular* if $\mu \perp \nu$ whenever $\mu, \nu \in \mathscr{F}$ and $\mu \neq \nu$.

The collection of such singular families in $M(K)^+$ is ordered by inclusion.

Let S be a non-empty subset of $M(K)^+$. It is clear from Zorn's lemma that the collection of singular families contained in S has a maximal member that contains any specific singular family in S; this is a *maximal singular family in S*. In the case where $S = P(K)$, we may suppose that such a maximal singular family contains all the measures that are point masses and that all other members are continuous measures, so that, in the case where K is discrete, the family of point masses is a maximal singular family in $P(K)$.

We shall see in Proposition 5.2.7 that any two infinite, maximal singular families of continuous measures have the same cardinality.

Proposition 4.6.2. (i) *Let K be a non-empty, locally compact space, and suppose that S is a separable subspace of $M(K)^+$. Then each singular family of measures in S is countable.*

(ii) *The space $M_c(\Delta)$ contains a singular family in $P(\Delta)$ of cardinality $\mathfrak{c}$.*

(iii) *Let K be an uncountable, compact, metrizable space. Then there is a maximal singular family of measures in $P(K)$ consisting of exactly $\mathfrak{c}$ point masses and $\mathfrak{c}$ continuous measures.*

Proof. (i) Let $\mathscr{F}$ be a singular family of measures in S. For each $\mu, \nu \in \mathscr{F}$ with $\mu \neq \nu$, we have $\|\mu - \nu\| = \|\mu\| + \|\nu\|$. For $n \in \mathbb{N}$, set $\mathscr{F}_n = \{\mu \in \mathscr{F} : \|\mu\| > 1/n\}$. For $\mu, \nu \in \mathscr{F}_n$ with $\mu \neq \nu$, we have $\|\mu - \nu\| > 2/n$, and so the open balls $B_{1/n}(\mu)$ and $B_{1/n}(\nu)$ are disjoint. Since S is separable, it follows that $\mathscr{F}_n$ is countable for each $n \in \mathbb{N}$, and so $\mathscr{F}$ is countable.

(ii) The Cantor cube $L = \mathbb{Z}_2^{\omega}$, identified with Δ, is a compact group and so has a Haar measure, say m_L, as on page 112, and $m_L \in M_c(L)$. By Proposition 1.4.5, L contains $\mathfrak{c}$ pairwise-disjoint, closed subspaces, each homeomorphic to L. We may transfer a copy of m_L to each of these subspaces; the resulting measures are mutually singular.

(iii) By Proposition 1.4.14, K contains Δ as a closed subspace. Let $\mathscr{F}$ be a maximal singular family of measures in $P(K)$ containing the family specified in (ii), so that $\mathscr{F}$ contains at least $\mathfrak{c}$ continuous measures. By Proposition 4.2.3, $|M(K)| = \mathfrak{c}$, and so $|\mathscr{F}| \leq \mathfrak{c}$. By Corollary 1.4.15, $|K| = \mathfrak{c}$, and hence $\mathscr{F}$ contains exactly $\mathfrak{c}$ point masses. □

We note that, under some mild set-theoretic axioms, such as Martin's axiom, there exists a compact space K with $|K| = \mathfrak{c}$ such that there is a maximal singular family in $P(K)$ of cardinality $2^{\mathfrak{c}}$: see [108].

Lemma 4.6.3. *Let K be a non-empty, locally compact space, and let $\mathscr{F}$ be a maximal singular family in $P(K)$. Then, for each $\nu \in M(K)$, there exist a countable subset Γ of $\mathscr{F}$ and $\nu_\mu \in M(K)$ for each $\mu \in \Gamma$ such that $\nu_\mu \ll \mu$ $(\mu \in \Gamma)$, such that $\nu = \sum\{\nu_\mu : \mu \in \Gamma\}$, and such that*

$$\|\nu\| = \sum\left\{\|\nu_\mu\| : \mu \in \Gamma\right\}.$$

The correspondence $\nu \mapsto (\nu_\mu)$, $M(K) \to M(K)^{\mathscr{F}}$, is a lattice homomorphism.

Proof. Take $\nu \in M(K)$. By the Lebesgue decomposition theorem, Theorem 4.2.9, for each $\mu \in \mathscr{F}$, there exist $\nu_\mu \ll \mu$ and $\sigma_\mu \perp \mu$ such that $\nu = \nu_\mu + \sigma_\mu$. Set $\Gamma = \{\mu \in \mathscr{F} : \nu_\mu \neq 0\}$.

For distinct elements $\mu_1, \dots, \mu_n \in \mathscr{F}$, we have $\mu_i \perp \mu_j$ whenever $i, j \in \mathbb{N}_n$ with $i \neq j$, and so $\nu = \nu_{\mu_1} + \cdots + \nu_{\mu_n} + \sigma$ for some $\sigma \in M(K)$ with $\sigma \perp \nu_{\mu_i}$ $(i \in \mathbb{N}_n)$, and then $\sum_{i=1}^n \left\|\nu_{\mu_i}\right\| \leq \|\nu\|$. It follows that Γ is countable, that we can define $\rho = \sum\{\nu_\mu : \mu \in \Gamma\}$ in $M(K)$, and that $\sum\{\left\|\nu_\mu\right\| : \mu \in \Gamma\} \leq \|\nu\|$.

Clearly $|\nu - \rho| \perp \mu$ for each $\mu \in \mathscr{F}$, and so $\nu - \rho = 0$ by the maximality of $\mathscr{F}$. Thus $\nu = \sum\{\nu_\mu : \mu \in \Gamma\}$, and so $\|\nu\| \leq \sum\{\left\|\nu_\mu\right\| : \mu \in \Gamma\}$.

It follows that $\|\nu\| = \sum\{\left\|\nu_\mu\right\| : \mu \in \Gamma\}$.

Clearly, the correspondence $\nu \mapsto (\nu_\mu)$, $M(K) \to M(K)^{\mathscr{F}}$, is a lattice homomorphism. □

Let K be a non-empty, locally compact space, and take $\mu \in P(K)$. As in Definition 4.4.5, Φ_μ denotes the character space of the C^*-algebra $L^\infty(K, \mu)$.

Definition 4.6.4. Let K be a non-empty, locally compact space, let S be a non-empty subset of $P(K)$, and let $\mathscr{F}$ be a maximal singular family in S. Define $U_{\mathscr{F}}$ to be the space that is the disjoint union of the sets Φ_μ for $\mu \in S$, with the topology in which each Φ_μ is a compact and open subspace of $U_{\mathscr{F}}$.

We now give our first representation of the Banach space $M(K)' = C_0(K)''$.

Theorem 4.6.5. *Let K be a non-empty, locally compact space, and let $\mathscr{F}$ be a maximal singular family in $P(K)$. Then*

$$\|\Lambda\| = \sup\{|\langle \Lambda, \nu\rangle| : \nu \ll \mu, \|\nu\| \leq 1, \mu \in \mathscr{F}\} \quad (\Lambda \in M(K)'), \tag{4.14}$$

and $M(K)' \cong C^b(U_{\mathscr{F}})$.

Proof. Set $U = U_{\mathscr{F}}$.

Take $\Lambda \in M(K)'$, say with $\|\Lambda\| = 1$. For each $\mu \in \mathscr{F}$, set $\Lambda_\mu = \Lambda \mid L^1(K, \mu)$, so that $\Lambda_\mu \in L^1(K, \mu)' = C(\Phi_\mu)$ with $\left\|\Lambda_\mu\right\| \leq 1$. Hence there exists $F_\mu \in C(\Phi_\mu)$ with $\left|F_\mu\right|_{\Phi_\mu} \leq 1$ and

$$\langle \rho, F_\mu\rangle = \langle \rho, \Lambda\rangle \quad (\rho \in L^1(K, \mu)).$$

Now define $F \in C^b(U)$ by requiring that $F \mid \Phi_\mu = F_\mu$ $(\mu \in \mathscr{F})$; set $\alpha = |F|_U$, so that $\alpha \leq 1$.

Take $\nu \in M(K)_{[1]}$. By Lemma 4.6.3, there is a countable subset Γ of $\mathscr{F}$ and $\nu_\mu \in M(K)$ for each $\mu \in \Gamma$ such that $\nu_\mu \ll \mu$ $(\mu \in \Gamma)$, such that $\nu = \sum\{\nu_\mu : \mu \in \Gamma\}$, and such that $\|\nu\| = \sum\{\left\|\nu_\mu\right\| : \mu \in \Gamma\}$. We have

$$|\langle \Lambda, \nu\rangle| = \left|\sum\{\langle \Lambda, \nu_\mu\rangle : \mu \in \Gamma\}\right| \leq \sum\{\left|\langle F_\mu, \nu_\mu\rangle\right| : \mu \in \Gamma\} \leq \alpha,$$

and so $1 \leq \alpha$. Thus $|F|_U = \|\Lambda\|$. Set $T(\Lambda) = F$, so that $T : M(K)' \to C^b(U)$ is an isometric linear map.

Conversely, given $F \in C^b(U)$, set $F_\mu = F \mid \Phi_\mu$ $(\mu \in \mathscr{F})$. For each $\nu \in M(K)$, write $\nu = \sum\{\nu_\mu : \mu \in \Gamma\}$, as before, and define

$$\Lambda(\nu) = \sum\{\langle F_\mu, \nu_\mu\rangle : \mu \in \mathscr{F}\}.$$

Then $\Lambda \in M(K)'$ and $T(\Lambda) = F$. It follows that T is a surjection, and so we have shown that $M(K)' \cong C^b(U)$.

To obtain equation (4.14), take $\Lambda \in M(K)'$ and $\varepsilon > 0$. Then there exists a measure $\mu \in \mathscr{F}$ such that $\left|T(\Lambda) \mid \Phi_\mu\right|_{\Phi_\mu} > \|\Lambda\| - \varepsilon$, and also there exists $\nu \in L^1(K,\mu)_{[1]}$ with $|\langle \Lambda, \nu\rangle| > \|\Lambda\| - \varepsilon$. Since $\nu \ll \mu$, equation (4.14) follows. □

Theorem 4.6.6. *Let K be an uncountable, compact, metrizable space. Then there are an index set J with $|J| = \mathfrak{c}$, measures $\mu_j \in P_c(K)$ for each $j \in J$, and a set Γ with $|\Gamma| = \mathfrak{c}$ such that*

$$M_c(K) \cong \bigoplus_1 \{L^1(K,\mu_j) : j \in J\} \cong \bigoplus_1 \{L^1(\mathbb{I})_j : j \in J\} \tag{4.15}$$

and

$$M(K) \cong \bigoplus_1 \{L^1(\mathbb{I})_j : j \in J\} \oplus_1 \ell^1(\Gamma), \tag{4.16}$$

where $L^1(\mathbb{I})_j = L^1(\mathbb{I})$ for each $j \in J$. Further, all the above identifications are Banach-lattice isometries.

Proof. By Proposition 4.6.2(iii), there is a maximal singular family, say $\{\mu_j : j \in J\}$, where $|J| = \mathfrak{c}$, of measures in $P_c(K)$. Set

$$E = \bigoplus_1 \{L^1(K,\mu_j) : j \in J\}.$$

Clearly E is a closed subspace of $M_c(K)$. Take $\mu \in M_c(K)$. For each $j \in J$, there exist $\rho_j, \sigma_j \in M_c(K)$ with $\rho_j \ll \mu$ and $\sigma_j \perp \mu$; we can regard each ρ_j as an element of $L^1(\mu_j)$. It follows from Lemma 4.6.3 that $\mu = \sum_{j\in J} \rho_j$, with $\|\mu\| = \sum_{j\in J} \left\|\rho_j\right\|$, so that $\mu \in E$. Thus $M_c(K) \cong \bigoplus_1 \{L^1(K,\mu_j) : j \in J\}$; the identification is a Banach-lattice isometry.

For each $j \in J$, the space $L^1(K,\mu_j)$ is separable, and so, by Theorem 4.4.14, $L^1(\mu_j)$ is Banach-lattice isometric to $L^1(\mathbb{I},m)$. Equation (4.15) follows.

Again by Proposition 4.6.2(iii), a maximal singular family in $P(K)$ is the set $\{\mu_j : j \in J\} \cup \{\delta_x : x \in K\}$, and so equation (4.16) follows, where we set $\Gamma = K$, so that $|\Gamma| = \mathfrak{c}$ by Proposition 1.4.14. □

Corollary 4.6.7. *Let K and L be two uncountable, compact, metrizable spaces. Then $M(K)$ and $M(L)$ are Banach-lattice isometric.*

Proof. This is immediate from equation (4.16). □

A generalization of Theorem 4.6.6 for an arbitrary measure space is given in *Maharam's theorem* [182], which is discussed in [166, §14] and [225, §26].

Theorem 4.6.8. *Let K be a non-empty, locally compact space, and suppose that $\{\mu_j : j \in J\}$ is a singular family in $P_c(K)$ with J uncountable. Then there is no embedding of the Banach space*

$$\bigoplus_1 \{L^1(K,\mu_j) : j \in J\}$$

into a Banach space of the form $F \oplus_1 \ell^1(D)$ for any separable Banach space F and any set D.

Proof. Let D be an index set, and take G to be the Banach space $(\ell^1(D), \|\cdot\|_1)$, and let F be a separable Banach space.

We shall apply Proposition 2.2.31. For each $j \in J$, the Banach space $L^1(K,\mu_j)$ contains an isometric copy of $L^1(\mathbb{I})$ by Theorem 4.4.14, and so, by Corollary 4.4.18, there is no embedding of $L^1(K,\mu)$ into $G = \ell^1(D)$. Thus, by Proposition 2.2.31, there is no embedding of $\bigoplus_1 \{L^1(K,\mu_j) : j \in J\}$ into $F \oplus_1 \ell^1(D)$. □

Corollary 4.6.9. *Let K be an uncountable, compact, metrizable space. Then the spaces $M_c(K)$ and $M(K)$ are not isomorphic to any closed subspace of a space of the form $F \oplus_1 \ell^1(D)$, where F is a separable Banach space and D is any set.*

Proof. Let $M_c(K)$ and $M(K)$ have the forms specified in equations (4.15) and (4.16), respectively. By Theorem 4.6.8, there is no isomorphism from the space $\bigoplus_1 \{L^1(K,\mu_j) : j \in J\}$ into $F \oplus_1 \ell^1(D)$, and so there is no such isomorphism from either $M_c(K)$ or $M(K)$. □

4.7 Normal measures

Let K be a non-empty, locally compact space. In this section, we shall introduce the (complex) Banach lattice $N(K)$ that consists of the normal measures on K, and we shall give a variety of examples of compact spaces K such that $N(K) = \{0\}$ and such that $N(K) \neq \{0\}$. A 'normal measure' was defined by Dixmier [91] to be an order-continuous measure $\mu \in M(K)$. Thus we have the following definition.

Definition 4.7.1. Let K be a non-empty, locally compact space, and let $\mu \in M(K)$. Then μ is *normal* if $\langle f_\alpha, \mu \rangle \to 0$ for each net $(f_\alpha : \alpha \in A)$ in $(C_0(K)^+, \leq)$ with $f_\alpha \searrow 0$ in the lattice, and μ is *σ-normal* if μ is σ-order-continuous, in the sense that$\langle f_n, \mu \rangle \to 0$ for each sequence $(f_n : n \in \mathbb{N})$ in $(C_0(K)^+, \leq)$ with $f_n \searrow 0$.

Definition 4.7.2. Let K be a non-empty, locally compact space. The subset of $M(K)$ consisting of the normal measures is $N(K)$; the set of real-valued measures in $N(K)$ is $N_{\mathbb{R}}(K)$, and the set of positive measures in $N(K)$ is $N(K)^+$. The sets of continuous and discrete normal measures on K are denoted by $N_c(K)$ and $N_d(K)$, respectively; further, we set $N_c(K)^+ = N_c(K) \cap M(K)^+$ and $N_d(K)^+ = N_d(K) \cap M(K)^+$.

It follows easily that $N(K)$, $N_d(K)$, and $N_c(K)$ are closed linear subspaces of $M(K)$. The point mass at an isolated point of K is a discrete normal measure.

The following proposition was proved in [91] and in detail by Bade in [24]. At certain points these sources require that the space K be Stonean; this is also the assumption in [234, Proposition III.1.11]. However, this assumption is not necessary.

Proposition 4.7.3. *Let K be a non-empty, locally compact space. Then:*

(i) *$\mu \in M(K)$ is normal if and only if $\Re\mu$ and $\Im\mu$ are normal;*

(ii) *$\mu \in M_{\mathbb{R}}(K)$ is normal if and only if $|\mu|$ is normal if and only if μ^+ and μ^- are normal;*

(iii) *$\mu \in M(K)$ is normal if and only if $|\mu|$ is normal;*

(iv) *$N(K)$ is a lattice ideal in $M(K)$, and $N(K) = N_d(K) \oplus_1 N_c(K)$.*

Proof. (i) This is immediate.

(ii) Suppose that $\mu^+, \mu^- \in N(K)$. Then certainly $\mu, |\mu| \in N(K)$.

Suppose that $|\mu| \in N(K)$ and that $\nu \in M(K)$ with $|\nu| \le |\mu|$. Then

$$0 \le \left| \int_K f_\alpha \,\mathrm{d}\nu \right| \le \int_K f_\alpha \,\mathrm{d}|\mu| \to 0 \tag{4.17}$$

when $f_\alpha \searrow 0$ in $C_0(K)^+$, and so $\nu \in N(K)$. In particular, μ, μ^+, and μ^- are normal whenever $|\mu|$ is normal.

Suppose that $\mu \in M_{\mathbb{R}}(K)$ is normal and that $f_\alpha \searrow 0$ in $C_0(K)^+_{[1]}$. Let $\{P,N\}$ be a Hahn decomposition of K with respect to μ, as in Theorem 4.1.7(i), and take $\varepsilon > 0$. Since μ is regular, there exist a compact set L and an open set U in K with $L \subset P \subset U$ and $|\mu|(U \setminus L) < \varepsilon$. By Theorem 1.4.25, there exists $g \in C_{00}(K)^+$ with $\chi_L \le g \le \chi_U$. Then

$$\int_K f_\alpha \,\mathrm{d}\mu^+ = \int_P f_\alpha \,\mathrm{d}\mu \le \int_L g f_\alpha \,\mathrm{d}\mu + \int_{U\setminus L} g f_\alpha \,\mathrm{d}\mu + 2\varepsilon = \int_K g f_\alpha \,\mathrm{d}\mu + 2\varepsilon\,.$$

Since $g f_\alpha \searrow 0$ and μ is normal, $\lim_\alpha \langle g f_\alpha, \mu \rangle = 0$, and so

$$\limsup_\alpha \,\langle f_\alpha, \mu^+ \rangle \le 2\varepsilon\,.$$

This holds true for each $\varepsilon > 0$, and so $\lim_\alpha \langle f_\alpha, \mu^+ \rangle = 0$. Thus μ^+ is normal; similarly, μ^- is normal.

(iii) Suppose that $\mu \in N(K)$. Then $|\Re\mu| + |\Im\mu| \in N(K)$ from (i) and (ii). However $|\mu| \le |\Re\mu| + |\Im\mu|$, and so $|\mu| \in N(K)$.

(iv) This is immediate from (4.17). □

Note that $\lambda\mu \in N(K)$ for each $\lambda \in L^\infty(\mu)$ and $\mu \in N(K)^+$, and so we may regard $L^\infty(K,\mu)$ as a closed subspace of $N(K)$ for each $\mu \in N(K)^+$. In particular, the restriction of a normal measure on K to a Borel subspace of K is still a normal measure in the space $N(K)$.

The spaces of σ-normal measures on K have analogous properties to those in Proposition 4.7.3.

Let K be a locally compact space. Recall from Definition 1.4.1 that $\mathscr{K}_K$ denotes the family of compact subsets L of K such that $\mathrm{int}_K L = \emptyset$. Clause (i) of the following theorem, for Stonean spaces K, is due to Dixmier [91]; see [225, p. 341]. Clause (ii) was formulated and proved in [76, p. 405].

Theorem 4.7.4. *Let K be a non-empty, locally compact space. Then:*

(i) *a measure $\mu \in M(K)$ is normal if and only if $\mu(L) = 0 \ (L \in \mathscr{K}_K)$;*

(ii) *a measure $\mu \in M(K)$ is σ-normal if and only if $\mu(L) = 0$ for each G_δ-set $L \in \mathscr{K}_K$.*

Proof. (i) Suppose that $\mu \in N(K)$. By Proposition 4.7.3(iii), we may suppose that $\mu \in N(K)^+$. Now take $L \in \mathscr{K}_K$, and consider the non-empty set

$$\mathscr{F} = \{f \in C_{\mathbb{R}}(K) : f \geq \chi_L\}.$$

Suppose that $g = \inf \mathscr{F}$ in $C_{0,\mathbb{R}}(K)$. Then $g(x) = 0 \ (x \in K \setminus L)$, and so $g = 0$ because $\mathrm{int}_K L = \emptyset$. Thus $\inf \mathscr{F} = 0$. Since $\mu(L) = \inf\{\langle f, \mu\rangle : f \in \mathscr{F}\}$, we have $\mu(L) = 0$.

Conversely, suppose that $\mu \in M(K)$ and $\mu(L) = 0 \ (L \in \mathscr{K}_K)$. Again by Proposition 4.7.3(iii), it suffices to suppose that $\mu \in M(K)^+$. Take (f_α) in $C_0(K)^+$ with $f_\alpha \searrow 0$; we may suppose that $f_\alpha \leq 1$ for each α. Set

$$g(x) = \inf_\alpha f_\alpha(x) \quad (x \in K).$$

Then g is a Borel function because $g^{-1}(V)$ is an F_σ-set in K for each open set V in $\mathbb{R}$, and $g \geq 0$. For $n \in \mathbb{N}$, set $B_n = \{x \in K : g(x) > 1/n\}$, so that $B_n \in \mathfrak{B}_K$. For each compact subset L of B_n, we have $\mathrm{int}_K L = \emptyset$, and so $\mu(L) = 0$. Thus $\mu(B_n) = 0$, and so $\mu(\{x \in K : g(x) > 0\}) = 0$, whence $\int_K g \, \mathrm{d}\mu = 0$. Hence it suffices to show that

$$\lim_\alpha \int_K f_\alpha \, \mathrm{d}\mu = \int_K g \, \mathrm{d}\mu. \tag{4.18}$$

Take $\varepsilon > 0$. By Lusin's theorem, Proposition 4.1.7(ii), there is a compact subset L of K with $\mu(K \setminus L) < \varepsilon$ and such that $g \mid L \in C(L)$. By Dini's theorem, Theorem 1.4.28, $\lim_\alpha |f_\alpha \mid L - g \mid L|_L = 0$, and so there exists α_0 with $|f_\alpha \mid L - g \mid L|_L < \varepsilon \ (\alpha \geq \alpha_0)$. It follows that

$$\left| \int_K f_\alpha \, \mathrm{d}\mu - \int_K g \, \mathrm{d}\mu \right| < \int_L |f_\alpha - g| \, \mathrm{d}\mu + 2\varepsilon < (\|\mu\| + 2)\varepsilon \quad (\alpha \geq \alpha_0),$$

giving (4.18).

(ii) This is similar. $\square$

Consider Lebesgue measure m on $\mathbb{I}$. There are Cantor-type closed subsets L of $\mathbb{I}$ such that $\mathrm{int}\, L = \emptyset$ and $m(L) > 0$. This shows that m is not a σ-normal measure.

Corollary 4.7.5. *Let K be a non-empty, locally compact space, and suppose that $\mu \in M(K)$. Then the following are equivalent:*

(a) $\mu \in N(K)$;

(b) $|\mu|(\overline{B} \setminus \operatorname{int} B) = 0$ *for each* $B \in \mathfrak{B}_K$;

(c) $\mu(B_1) = \mu(B_2)$ *for each* $B_1, B_2 \in \mathfrak{B}_K$ *with* $B_1 \equiv B_2$.

Proof. We may suppose that $\mu \in M(K)^+$.

(a) $\Rightarrow$ (b) Take $B \in \mathfrak{B}_K$. For each $\varepsilon > 0$, there exists an open set U in K with $B \subset U$ and $\mu(U \setminus B) < \varepsilon$. Since $\overline{U} \setminus U \in \mathscr{K}_K$, we have $\mu(\overline{U} \setminus U) = 0$. Thus

$$\mu(B) \leq \mu(\overline{B}) \leq \mu(\overline{U}) = \mu(U) \leq \mu(B) + \varepsilon,$$

and so $\mu(\overline{B}) = \mu(B)$. By taking complements, it follows that $\mu(\operatorname{int} B) = \mu(B)$. Hence $\mu(\overline{B} \setminus \operatorname{int} B) = 0$.

(a) $\Rightarrow$ (c) We know that $\mu(B) = 0$ for each nowhere dense set B in $\mathfrak{B}_K$, and so $\mu(B) = 0$ for each meagre set B in $\mathfrak{B}_K$. Thus $\mu(B_1) = \mu(B_2)$ whenever $B_1, B_2 \in \mathfrak{B}_K$ with $B_1 \Delta B_2$ meagre.

(b), (c) $\Rightarrow$ (a) These are immediate from Theorem 4.7.4(i). □

Corollary 4.7.6. *Let K be a Stonean space, and suppose that $\mu \in N(K) \cap P(K)$ is a strictly positive measure. Then every equivalence class in $L^\infty(K,\mu)$ contains a continuous function, the C^*-algebras $(L^\infty(K,\mu), \|\cdot\|_\infty)$ and $(C(K), |\cdot|_K)$ are C^*-isomorphic, and Φ_μ is homeomorphic to K.*

Proof. By Theorem 3.3.5(iii), there is a C^*-isomorphism $\overline{P} : B^b(K)/M_K \to C(K)$. However $\mu(B) = 0$ for each meagre set $B \in \mathfrak{B}_K$ by Corollary 4.7.5, and so ker $\overline{P}$ is exactly the kernel of the projection of $B^b(K)$ onto $L^\infty(K,\mu)$. The result follows. □

Proposition 4.7.7. *Let K be a non-empty, locally compact space satisfying* CCC. *Then every σ-normal measure on K is normal.*

Proof. Let $\mu \in M(K)$ be σ-normal. We must show that $\mu \in N(K)$; it suffices to suppose that $\mu \in M(K)^+$. Recall from page 23 that $\mathbf{Z}(K)$ denotes the family of zero sets of K. By Theorem 4.7.4(ii), $\mu(Z) = 0$ for each $Z \in \mathscr{K}_K \cap \mathbf{Z}(K)$.

Take $L \in \mathscr{K}_K$. We *claim* that there exists $Z \in \mathscr{K}_K \cap \mathbf{Z}(K)$ such that $L \subset Z$. Indeed, let $\mathscr{F}$ be a maximal disjoint family of cozero sets contained in the open set $K \setminus L$. By CCC, $\mathscr{F}$ is countable, and so the set

$$Z := \bigcap \{K \setminus V : V \in \mathscr{F}\}$$

is a zero set containing L. Hence Z has empty interior by the maximality of $\mathscr{F}$, proving the claim.

By hypothesis, $\mu(Z) = 0$. Thus $\mu(L) = 0$, and so it follows from Theorem 4.7.4(i) that $\mu \in N(K)$. □

Consider the compact space $K := [0,\omega_1]$. Then $\delta_{\omega_1} \in M(K)^+$ and $\delta_{\omega_1}(Z) = 0$ for each $Z \in \mathscr{K}_K$ that is a zero set because each zero set that contains ω_1 has non-empty interior. Thus δ_{ω_1} is a σ-normal measure on K which is not normal (because $\{\omega_1\}$ is compact with empty interior). Another such example will be given below in Example 4.7.16.

We note that, if one asks whether such an example can be found on a Stonean space K, large cardinals come into the picture. The existence of a Stonean space K with a non-zero σ-normal measure which is not normal is equivalent to the existence of a measurable cardinal; see [107, Theorem 363S] or [179].

Theorem 4.7.8. *Let K be a non-empty, locally compact space. Then:*

(i) *$N(K)$ is a Dedekind complete lattice ideal in $M(K)$;*

(ii) *there is a closed subspace $S(K)$ of $M(K)$ such that $M(K) = N(K) \oplus_1 S(K)$ and $\nu \perp \sigma$ for each $\nu \in N(K)$ and $\sigma \in S(K)$;*

(iii) *$N(K)$ is a 1-complemented subspace of $M(K)$.*

Proof. (i) By Proposition 4.7.3(iv), $N(K)$ is a lattice ideal in $M(K)$.

Let $\mathfrak{F}$ be a family that is bounded above in $N(K)^+$, and set $\mu = \bigvee \mathfrak{F}$ in $M(K)^+$, so that

$$\mu(B) = \sup\{\nu(B) : \nu \in \mathfrak{F}\} \quad (B \in \mathfrak{B}_K)\,.$$

This implies that $\mu(L) = 0$ $(L \in \mathscr{K}_K)$, and so $\mu \in N(K)^+$; clearly, μ is the supremum of $\mathfrak{F}$ in $N(K)^+$, and so $N(K)$ is Dedekind complete.

(ii) Set

$$S(K) = \{\sigma \in M(K) : \nu \perp \sigma \ \ (\nu \in N(K))\}\,.$$

Then $S(K)$ is a closed linear subspace of $M(K)$ and $N(K) \cap S(K) = \{0\}$.

Now take $\mu \in M(K)^+$, and set

$$\mu_{\rm n} = \bigvee \{\nu \in N(K)^+ : \nu \leq \mu\}\,,$$

so that $\mu_{\rm n} \in N(K)^+$; set $\mu_{\rm s} = \mu - \mu_{\rm n}$. For $\nu \in N(K)^+$, we have $\mu_{\rm n} + (\mu_{\rm s} \wedge \nu) \leq \mu$, and hence $\mu_{\rm n} + (\mu_{\rm s} \wedge \nu) \leq \mu_{\rm n}$. Thus $\mu_{\rm s} \wedge \nu = 0$ $(\nu \in N(K)^+)$. It follows that $\mu_{\rm s} \in S(K)^+$.

For $\mu \in M(K)$, write $\mu = \mu_1 - \mu_2 + {\rm i}(\mu_3 - \mu_4)$, where $\mu_1, \dots, \mu_4 \in M(K)^+$. For $i = 1, \dots, 4$, the measure μ_i can be decomposed as $\mu_{i,{\rm n}} + \mu_{i,{\rm s}}$ with $\mu_{i,{\rm n}} \in N(K)^+$ and $\mu_{i,{\rm n}} \in S(K)^+$. Set

$$\mu_{\rm n} = \mu_{1,{\rm n}} - \mu_{2,{\rm n}} + {\rm i}(\mu_{3,{\rm n}} - \mu_{4,{\rm n}}) \quad \text{and} \quad \mu_{\rm s} = \mu_{1,{\rm s}} - \mu_{2,{\rm s}} + {\rm i}(\mu_{3,{\rm s}} - \mu_{4,{\rm s}})\,.$$

Then $\mu_{\rm n} \in N(K)$, $\mu_{\rm s} \in S(K)$, and $\mu = \mu_{\rm n} + \mu_{\rm s}$, so that $M(K) = N(K) \oplus S(K)$. Since $\mu_{\rm n} \perp \mu_{\rm s}$, we have $\|\mu\| = \|\mu_{\rm n}\| + \|\mu_{\rm s}\|$, and so $M(K) = N(K) \oplus_1 S(K)$.

(iii) This is immediate from (ii). □

The measures in $S(K)$ are sometimes called the *singular* measures, although this is a somewhat unfortunate term.

Proposition 4.7.9. *Let K be a non-empty, locally compact space, and suppose that $\mu \in N(K)$. Then* supp μ *is a regular–closed set.*

Proof. Since supp $\mu = $ supp $|\mu|$, we may suppose that $\mu \in N(K)^+$.

Set $F = $ supp μ, a closed set, and set $U = \operatorname{int} F$, so that $\overline{U} \subset F$. Since $F \setminus \overline{U}$ is nowhere dense, $\mu(F \setminus \overline{U}) = 0$ by Theorem 4.7.4(i). Thus $\mu(K \setminus \overline{U}) = 0$, and so, by the definition of supp μ, we have $K \setminus \overline{U} \subset K \setminus F$. Hence $\overline{U} = F$, and F is regular-closed. □

The next corollary does use the fact that K is Stonean; the result is due to Dixmier [91], and is set out by Bade in [23, Lemma 8.6].

Corollary 4.7.10. *Let K be a Stonean space, and suppose that $\mu \in N(K)^+ \setminus \{0\}$.*

(i) *The space* supp μ *is clopen in K, and hence Stonean.*

(ii) *For each $B \in \mathfrak{B}_K$, there is a unique set $C \in \mathfrak{C}_K$ with $C \subset$* supp μ *and $\mu(B \Delta C) = 0$, and so each equivalence class in $\mathfrak{B}_\mu$ contains a unique clopen subset of* supp μ.

Proof. (i) In a Stonean space, every regular–closed set is clopen.

(ii) By (i), supp μ is a clopen subset of K and $\mu(K \setminus \operatorname{supp}\mu) = 0$, and so we may suppose that $K = \operatorname{supp}\mu$.

Take $B \in \mathfrak{B}_K$. By Proposition 1.4.4, there is a unique $C \in \mathfrak{C}_K$ with $B \equiv C$, and then $\mu(B \Delta C) = 0$. Suppose that $C_1, C_2 \in \mathfrak{C}_K$ are such that $\mu(B \Delta C_1) = \mu(B \Delta C_2) = 0$. Then $C_1 \Delta C_2 \subset (B \Delta C_1) \cup (B \Delta C_2)$, so that $\mu(C_1 \Delta C_2) = 0$. Since $C_1 \Delta C_2$ is an open set and $K = \operatorname{supp}\mu$, it follows from Proposition 4.1.6 that $C_1 \Delta C_2 = \emptyset$, i.e., $C_1 = C_2$. This establishes the required uniqueness of C. □

Corollary 4.7.11. *Let K be a Stonean space, and suppose that $\mu, \nu \in N(K)$. Then:*

(i) supp $\nu \subset$ supp μ *if and only if* $\nu \ll \mu$ *;*

(ii) supp $\nu =$ supp μ *if and only if* $\nu \sim \mu$ *;*

(iii) $\mu \perp \nu$ *if and only if* supp $\mu \cap$ supp $\nu = \emptyset$.

Proof. (i) Always supp $\nu \subset$ supp μ when $\nu \ll \mu$.

For the converse, we may suppose that $\mu, \nu \in N(K)^+$. By Proposition 1.4.4, for each $B \in \mathfrak{B}_\mu$, there exists $C \in \mathfrak{C}_K$ with $C \equiv B$. Now suppose that $B \in \mathfrak{N}_\mu$. Then, by Corollary 4.7.5(ii), $C \in \mathfrak{N}_\mu$, and so $C \cap$ supp $\nu = \emptyset$, whence $\nu(B) = \nu(C) = 0$. This shows that $\nu \ll \mu$.

(ii) This is immediate from (i).

(iii) Clearly $\mu \perp \nu$ when supp $\mu \cap$ supp $\nu = \emptyset$.

Now suppose that $\mu \perp \nu$, and set $U =$ supp $\mu \cap$ supp ν, so that, by Corollary 4.7.10(i), U is an open set. Then $(\nu \mid U) \perp \mu$ and, by (i), $\nu \mid U \ll \mu$. Thus $\nu \mid U = 0$, and hence $U = \emptyset$. □

We now determine the set of extreme points of the closed unit ball of the normal measures. Recall that D_X denotes the set of isolated points of a topological space X.

Proposition 4.7.12. *Let K be a non-empty, locally compact space. Then*

$$\operatorname{ex} N(K)_{[1]} = \{\zeta \delta_x : \zeta \in \mathbb{T}, x \in D_K\} \quad \text{and} \quad \operatorname{ex} N(K) \cap P(K) = \{\delta_x : x \in D_K\}.$$

Proof. By Proposition 2.1.10 and Theorem 4.7.8(ii),

$$\operatorname{ex} M(K)_{[1]} = \operatorname{ex} N(K)_{[1]} \cup \operatorname{ex} S(K)_{[1]}\,.$$

Thus, by Proposition 4.4.15(i), each point of $\operatorname{ex} N(K)_{[1]}$ has the form $\zeta\delta_x$ for some $\zeta \in \mathbb{T}$ and $x \in K$. By Theorem 4.7.4(i), $\operatorname{int}_K\{x\} \neq \emptyset$, and so $x \in D_K$.

Conversely, $\zeta\delta_x \in \operatorname{ex} N(K)_{[1]}$ whenever $\zeta \in \mathbb{T}$ and $x \in D_K$. □

Corollary 4.7.13. *Let K be a non-empty, locally compact space. Then we can identify $N_d(K)$ with $\ell^1(D_K)$ and $N_c(K)$ with $N(K \setminus \overline{D_K})$.*

Proof. We know that $\delta_x \in N_d(K)$ for each $x \in D_K$, and so $\ell^1(D_K) \subset N_d(K)$. Conversely, it is clear that $N_d(K) \subset \ell^1(D_K)$. Thus $N_d(K) = \ell^1(D_K)$.

For each $\mu \in N(K)$, we have $|\mu|(\overline{D_K} \setminus D_K) = 0$ by Corollary 4.7.5, and so we have $\operatorname{supp}\mu \subset K \setminus \overline{D_K}$ for each $\mu \in N_c(K)$. Conversely, take $\mu \in N(K \setminus \overline{D_K})$. Then $|\mu|(\{x\}) = 0$ $(x \in K \setminus D_K)$, and so $\mu \in N_c(K)$. □

Corollary 4.7.14. *Let S be a non-empty set. Then $N(\beta S) = N_d(\beta S) = \ell^1(S)$ and $N_c(\beta S) = N(S^*) = \{0\}$.*

Proof. By Proposition 1.5.9(ii), βS is Stonean, and $\overline{D_{\beta S}} = \overline{S} = \beta S$. By Corollary 4.7.13, $N(\beta S) = N_d(\beta S) = \ell^1(S)$ and $N_c(\beta S) = \{0\}$.

We now show that $N(S^*) = \{0\}$. Assume to the contrary that $\mu \in N(S^*)$ with $\mu \neq 0$. By Theorem 4.7.4(i), $\operatorname{supp}\mu$ has non-empty interior, and so $\operatorname{supp}\mu$ contains a clopen set of the form A^*, where A is an infinite subset of S. By Proposition 1.5.5, A^* contains an uncountable family of non-empty, pairwise-disjoint, open subsets. But this contradicts the fact that, by Proposition 4.1.6, $\operatorname{supp}\mu$ satisfies CCC. Thus $\mu = 0$. □

Corollary 4.7.15. *Let X be a non-empty, compact space such that $N(X)$ is isometrically a dual space. Suppose that D_X is countable and infinite. Then $N(X) \cong \ell^1$.*

Proof. Take E to be a Banach space with $E' \cong N(X)$; we shall apply Theorem 4.1.10 with K taken to be $E'_{[1]}$. Take a countable, dense subset T of $\mathbb{T}$, and consider the countable set

$$D = \{\zeta\delta_x : \zeta \in T, x \in D_X\}\,.$$

Then, using Proposition 4.7.12, we see that D is $\|\cdot\|$-dense in $\operatorname{ex} K$, and so, by Theorem 4.1.10, K is the $\|\cdot\|$-closure of the absolutely convex hull of $\{\delta_x : x \in D_X\}$. It follows that $E' \cong \ell^1$, and so $N(X) \cong \ell^1$. □

The next example gives some σ-normal measures on a space K that is such that $N(K) = \{0\}$.

Example 4.7.16. Consider the compact space $K = \mathbb{N}^*$. By Proposition 1.5.3(i), there are no non-empty G_δ-sets in $\mathscr{K}_K$. Thus all measures in $M(K)$ are σ-normal. However $N(K) = \{0\}$ by Corollary 4.7.14. □

Let K and L be non-empty, compact spaces, and again suppose that $\eta : K \to L$ is a continuous surjection. Recall that we defined

$$\eta^\circ : f \mapsto f \circ \eta\,, \quad C(L) \to C(K)\,,$$

in equation (2.9) on page 83, so that η° is a unital C^*-embedding and a lattice homomorphism. The dual of η° is therefore a surjection

$$T_\eta := (\eta^\circ)' : M(K) \to M(L)$$

with $\|T_\eta\| = 1$; of course, as in equation (4.7) on page 116,

$$(T_\eta \mu)(B) = \mu(\eta^{-1}(B)) \quad (B \in \mathfrak{B}_L, \mu \in M(K))\,, \tag{4.19}$$

and $T_\eta \mu$ is the image measure $\eta[\mu]$. We shall use this notation in the next result.

Note that $T_\eta \mu \in M(L)^+$ when $\mu \in M(K)^+$, and so T_η is a positive operator on the Banach lattice $M(K)$, and hence is an order homomorphism. (However, it is easily seen that T_η is not necessarily a lattice homomorphism.) Now take $\nu \in M(L)^+$. Then ν defines a positive linear functional on $\eta^\circ(C(L))$, and so has a norm-preserving extension to a linear functional on $C(K)$, and hence to a measure $\mu \in M(K)$ with $\|\mu\| = \|\nu\|$; by equation (4.2), $\mu \in M(K)^+$. In particular, this shows that $T_\eta(M(K)^+) = M(L)^+$.

Proposition 4.7.17. *Let K and L be non-empty, compact spaces, and suppose that $\eta : K \to L$ is a continuous surjection that is either open or irreducible. Then*

$$T_\eta(N(K)) \subset N(L)\,.$$

Suppose, further, that $N(L) = \{0\}$. Then $N(K) = \{0\}$.

Proof. Take $\mu \in N(K)$. For $L_0 \in \mathscr{K}_L$, set $K_0 = \eta^{-1}(L_0)$. Then K_0 is certainly compact in K. We *claim* that $\mathrm{int}_K K_0 = \emptyset$. This is obvious when η is open, and follows from Proposition 1.4.21(ii) when η is irreducible. Thus $K_0 \in \mathscr{K}_K$. By Theorem 4.7.4(i), $\mu(K_0) = 0$, and so $(T_\eta \mu)(L_0) = 0$. Again by Theorem 4.7.4(i), $T_\eta \mu \in N(L)$. Thus $T_\eta(N(K)) \subset N(L)$.

Now suppose that $N(L) = \{0\}$, and take $\mu \in N(K)^+$. Then $T_\eta \mu = 0$. But this implies that $\mu(K) = (T_\eta \mu)(L) = 0$, and hence $\mu = 0$. Thus $N(K) = \{0\}$. □

Theorem 4.7.18. *Let K and L be two non-empty, compact spaces, and suppose that $\eta : K \to L$ is an irreducible surjection. Then the map*

$$T_\eta \mid N(K) : N(K) \to N(L) \tag{4.20}$$

is a Banach-lattice isometry.

Proof. By Proposition 4.7.17, $T_\eta(N(K)) \subset N(L)$. We shall now show that the map $T_\eta : N(K) \to N(L)$ is a bijection.

Set

$$\eta^{-1}(\mathfrak{B}_L) = \{\eta^{-1}(B) : B \in \mathfrak{B}_L\},$$

so that $\eta^{-1}(\mathfrak{B}_L)$ is a subset of $\mathfrak{B}_K$.

We *claim* that each $C \in \mathfrak{B}_K$ is congruent to a set in $\eta^{-1}(\mathfrak{B}_L)$. First suppose that U is a non-empty, open set in K, and define $V = \{y \in L : F_y \subset U\}$, where $F_y = \eta^{-1}(\{y\})$ $(y \in L)$. By Proposition 1.4.21(ii), V is open in L and $\eta^{-1}(V)$ is a dense, open subset of U, and so $\eta^{-1}(V) \in \eta^{-1}(\mathfrak{B}_L)$ and $U \equiv \eta^{-1}(V)$. As on page 13, each $C \in \mathfrak{B}_L$ has the Baire property, and so there is an open set U in K with $C \equiv U$. The claim follows.

Now suppose that $\mu \in N(K)$ with $T_\eta \mu = 0$. Then $\mu(\eta^{-1}(B)) = 0$ $(B \in \mathfrak{B}_L)$, and so $\mu(C) = 0$ $(C \in \mathfrak{B}_K)$ by the claim and Corollary 4.7.5, (a) $\Rightarrow$ (c). Thus the map $T_\eta : N(K) \to N(L)$ is an injection.

We next *claim* that $T_\eta : N(K) \to N(L)$ is a surjection and that the map

$$T_\eta \mid N(K)^+ : N(K)^+ \to N(L)^+$$

is an isometry. Indeed, take $\nu \in N(L)^+$. As above, there exists $\mu \in M(K)^+$ with $\|\mu\| = \|\nu\|$ and $T_\eta \mu = \nu$. Take $K_0 \in \mathscr{K}_K$, and set $L_0 = \pi(K_0)$. By Proposition 1.4.22, $L_0 \in \mathscr{K}_L$, and so $\nu(L_0) = 0$. Thus $\mu(\pi^{-1}(L_0)) = 0$. Since $\mu \in M(K)^+$, it follows that $\mu(K_0) = 0$, and hence $\mu \in N(K)^+$ by Theorem 4.7.4(i). The claim follows.

We have shown that the map $T_\eta \mid N_{\mathbb{R}}(K) \to N_{\mathbb{R}}(L)$ is a bijection and that it is an order isomorphism, and so $T_\eta \mid N(K) : N(K) \to N(L)$ is a Banach-lattice isomorphism. By Proposition 2.3.5 and the above claim, it is a Banach-lattice isometry. □

Corollary 4.7.19. *Let L be a non-empty, compact space. Then the map*

$$T_{\pi_L} \mid N(G_L) : N(G_L) \to N(L)$$

is a Banach-lattice isometry. In particular, $N(G_L) \cong N(L)$.

Proof. As in Theorem 1.6.5, the map $\pi_L : G_L \to L$ is an irreducible surjection, and so this is a special case of the theorem. □

Later, we shall be concerned with compact spaces that have many normal measures, but first we shall give various examples of compact spaces that have no non-zero normal measures.

Proposition 4.7.20. *Let K be a non-empty, separable, locally compact space without isolated points. Then there are no non-zero σ-normal measures on K, and so $N(K) = \{0\}$.*

Proof. We first *claim* that each σ-normal measure μ on EK is a continuous measure. Indeed, take $x \in K$. Since the point x is not isolated, there is a countable subset, say $S = \{x_n : n \in \mathbb{N}\}$, of $K \setminus \{x\}$ such that S is dense in K. Choose a sequence (U_n) in $\mathscr{N}_x$

such that $\overline{U_1}$ is compact and such that $\overline{U_{n+1}} \subset U_n$ and $x_n \notin U_n$ for each $n \in \mathbb{N}$, and set $L = \bigcap \overline{U_n}$. Then L is a compact G_δ-set in K with $x \in L$, and $\mathrm{int}_K L = \emptyset$ because $L \cap S = \emptyset$. By Theorem 4.7.4(ii), $\mu(L) = 0$. This implies that $\mu(\{x\}) = 0$, and hence μ is continuous, as claimed.

Again, let $\{x_n : n \in \mathbb{N}\}$ be a dense subset of K. Fix $\varepsilon > 0$ and a compact subset L of K; take $g \in C_{0,\mathbb{R}}(K)$ with $g \geq \chi_L$ and $g(K) \subset \mathbb{I}$. For each $n \in \mathbb{N}$, take $U_n \in \mathscr{N}_{x_n}$ with $|\mu|(U_n) < \varepsilon/2^n$, choose $f_n \in C_{00}(K)$ with $\chi_{\{x_n\}} \leq f_n \leq \chi_{U_n}$, and set $g_n = g \wedge \bigvee_{j=1}^n f_j$, so that $g_n \nearrow g$ in $C_0(K)^+$. We have

$$\langle g_n, |\mu|\rangle \leq |\mu| \left(\bigcup_{k=1}^n U_k \right) \leq \sum_{k=1}^n |\mu|(U_k) < \varepsilon \quad (n \in \mathbb{N}).$$

Since $|\mu|$ is σ-normal, $\langle g_n, |\mu|\rangle \nearrow \langle g, |\mu|\rangle$ in $\mathbb{R}^+$, and so $|\mu|(L) \leq \langle g, |\mu|\rangle \leq \varepsilon$. This holds true for each $\varepsilon > 0$, and hence $|\mu|(L) = 0$. Thus $\mu = 0$.

This gives the result. □

It is natural to wonder whether $N(K) = \{0\}$ when the condition 'separable' in Proposition 4.7.20 is replaced by the weaker condition that K satisfies CCC. The example of Theorem 4.7.26, to be given below, will show that this is not the case.

Corollary 4.7.21. *There are no non-zero, σ-normal measures on $G_{\mathbb{I}}$, and hence $N(G_{\mathbb{I}}) = \{0\}$.*

Proof. As remarked within Example 1.7.16, $G_{\mathbb{I}}$ is an infinite, separable Stonean space without isolated points, and so this follows from the proposition. The result also follows from Proposition 1.7.13. □

Corollary 4.7.22. *Let G be a locally compact group that is not discrete. Then $N(G) = \{0\}$.*

Proof. Take $\mu \in N(G)^+$ and a compact subspace K of G. Then there is an infinite, clopen, σ-compact subgroup G_0 of G with $G_0 \supset K$. As in Theorem 4.4.2, there is a non-discrete, metrizable group H and a quotient map $\eta : G_0 \to H$; the map η is open. The space $\eta(K)$ is separable and has no isolated points, and so, by Proposition 4.7.20, $N(\eta(K)) = \{0\}$. By Proposition 4.7.17, $N(K) = \{0\}$, and so $\mu(K) = 0$. It follows that $N(G) = \{0\}$. □

The following result is essentially contained in [103].

Theorem 4.7.23. *Let K be a non-empty, locally connected, locally compact space without isolated points. Then $N(K) = \{0\}$.*

Proof. Assume that there exists $\mu \in N(K)^+$ with $\mu \neq 0$. Again, $\mu \in N_c(K)^+$.

For each $n \in \mathbb{N}$, let $\mathscr{F}_n$ be a family of non-empty, open subsets of K such that $\mathscr{F}_n$ is maximal with respect to the following properties:

(i) $\mu(U) < 1/n$ for each $U \in \mathscr{F}_n$; (ii) distinct sets in $\mathscr{F}_n$ are disjoint.

It is clear from Zorn's lemma that such a family $\mathscr{F}_n$ exists. Set $G_n = \bigcup\{U : U \in \mathscr{F}_n$, an open subset of K. Since μ is continuous, each open set in K contains an open set of arbitrary small μ-measure, and so $\overline{G_n} = K$. By Theorem 4.7.4(i), $\mu(K \setminus G_n) = 0$.

Now set $H = \bigcap\{G_n : n \in \mathbb{N}\}$, a G_δ-set in K. We have $\mu(K \setminus H) = 0$, and so $\mu(H) > 0$. By Theorem 4.7.4(i), $\mu(\mathrm{int}_K H) > 0$. Assume that each $x \in \mathrm{int}_K H$ has an open neighbourhood V_x in K with $\mu(V_x) = 0$. For each compact subset L of $\mathrm{int}_K H$, there are finitely many points $x_1, \dots, x_n \in \mathrm{int}_K H$ with $L \subset V_{x_1} \cup \cdots \cup V_{x_n}$, and so $\mu(L) = 0$. But

$$\mu(\mathrm{int}_K H) = \sup\{\mu(L) : L \text{ compact}, \ L \subset \mathrm{int}_K H\}$$

because μ is a regular measure, and so $\mu(\mathrm{int}_K H) = 0$, a contradiction. Thus there exists $x_0 \in \mathrm{int}_K H$ such that $\mu(V) > 0$ for each $V \in \mathscr{N}_{x_0}$. Let V_0 be an open neighbourhood of x_0 with $V_0 \subset \mathrm{int}_K H$. Since K is locally connected, we may suppose that V_0 is connected. We have $V_0 \subset G_n$ for each $n \in \mathbb{N}$.

Since $\mu(V_0) > 0$, there exists $n \in \mathbb{N}$ with $\mu(V_0) > 1/n$. Choose $U \in \mathscr{F}_n$ with $x_0 \in U$, and set $V = G_n \setminus U$, so that V is open in K. Since $\mu(U) < 1/n < \mu(V_0)$, we have $V_0 \cap V \neq \emptyset$, and so $\{V_0 \cap U, V_0 \cap V\}$ is a partition of V_0 into two non-empty, disjoint, open subsets, a contradiction of the fact that V_0 is connected.

Thus $N(K) = \{0\}$, as required. □

Proposition 4.7.24. *Let K be a non-empty, connected, locally compact F-space. Then $N(K) = \{0\}$.*

Proof. Assume that there exists $\mu \in N(K)^+ \setminus \{0\}$, and choose a compact subset L of K such that $\mu(L) > 0$. Since L is a compact F-space satisfying CCC (by Proposition 4.1.6), the space L is Stonean, and so there is a non-empty, open subset U of K with $U \subset L$. Choose a non-empty, open subset V of K such that $\overline{V} \subset U$. Then $\overline{V}$ is open in U, and hence in K. We have shown that K contains a non-empty, clopen subset, and so K is not connected, the required contradiction. □

Proposition 4.7.25. *Let L be a compact space without isolated points which is either separable or a locally compact group or locally connected or a connected F-space, and suppose that K is a compact space such that there is a continuous surjection that is open or irreducible from K onto L. Then $N(K) = \{0\}$. In particular, $N(G_L) = \{0\}$ and $N(L \times R) = \{0\}$ for each compact space R.*

Proof. This follows from Proposition 4.7.17, Proposition 4.7.20, Corollary 4.7.22, Theorem 4.7.23, and Proposition 4.7.24. □

In the text [220, p. 2], a monotone complete C^*-algebra is said to be *wild* if there are no non-zero normal states. Let K be a non-empty, compact space. Then, as we remarked on page 107, $C(K)$ is a monotone complete C^*-algebra if and only if K is Stonean; $C(K)$ is wild if and only if $N(K) = \{0\}$. In [220, §4.3], it is shown that there are many examples of monotone complete C^*-subalgebras of ℓ^∞ that are wild, and so we obtain many examples of Stonean spaces K such that $N(K) = \{0\}$.

In the light of Theorem 4.7.23 and Proposition 4.7.24, it is natural to wonder whether $N(K) = \{0\}$ for each connected, compact set K. This question was answered by Grzegorz Plebanek [206] with the following counter-example; we are very grateful to him for his permission to include it here. Preliminary results on inverse systems with measures were given in §4.1.

Theorem 4.7.26. *There is a non-empty, connected, compact set K satisfying* CCC, *and such that $N(K) \neq \{0\}$. Indeed, there exists a strictly positive measure in $N(K)$.*

Proof. Let $L = \mathbb{I}$, a connected, compact space, and take m to be the strictly positive measure on $\mathbb{I}$ that is Lebesgue measure.

We shall define inductively an inverse system with strictly positive measures

$$(K_\alpha, \mu_\alpha, \pi_\alpha^\beta : 0 \le \alpha \le \beta < \omega_1)$$

with $K_0 = L$ and $\mu_0 = m$.

When $0 \le \gamma < \omega_1$ is such that $(K_\alpha, \mu_\alpha, \pi_\alpha^\beta : 0 \le \alpha \le \beta \le \gamma)$ is an inverse system with non-empty, connected, compact spaces K_α and strictly positive measures $\mu_\alpha \in P(K_\alpha)$ (for $0 \le \alpha \le \gamma$), we define $K_{\gamma+1}$ and $\mu_{\gamma+1}$ by applying Theorem 4.1.16 with $L = K_\gamma$ and $\nu = \mu_\gamma$ and by setting $K_{\gamma+1} = K_\gamma^\#$ and $\mu_{\gamma+1} = \mu_\gamma^\#$ (and defining the maps $\pi_\alpha^{\gamma+1}$ to be $\eta^\# \circ \pi_\alpha^\gamma$ for $0 \le \alpha \le \gamma$ and $\pi_{\gamma+1}^{\gamma+1}$ to be the identity on $K_{\gamma+1}$).

As in Theorem 4.1.16, we have $\operatorname{int}_{K_{\gamma+1}} (\pi_\gamma^{\gamma+1})^{-1}(W) \neq \emptyset$ for each $W \in \mathbf{Z}(K_\gamma)$ with $\mu_\gamma(W) > 0$.

When $0 \le \gamma \le \omega_1$, γ is a limit ordinal, and K_α and $\mu_\alpha \in P(K_\alpha)$ are defined for $0 \le \alpha < \gamma$, we define $(K_\gamma, \pi_\alpha^\gamma : 0 \le \alpha < \gamma)$ to be the inverse limit of the inverse system $(K_\alpha, \pi_\alpha^\beta : 0 \le \alpha \le \beta < \gamma)$ (and take π_α^γ to be the continuous surjections that arise in Theorem 1.4.32), so that K_γ is compact and connected; we take $\mu_\gamma \in P(K_\gamma)$ to be the strictly positive measure specified in Proposition 4.1.15. In the special case in which $\gamma = \omega_1$, we set $K = K_\gamma$, $\mu = \mu_\gamma \in P(K)$, and $\eta = \pi_0^\gamma$.

It follows from Corollary 1.4.33 that, for each $Z \in \mathbf{Z}(K)$, there exists $\alpha < \omega_1$ and $W \in \mathbf{Z}(K_\alpha)$ such that $Z = \pi_\alpha^{-1}(W)$. Suppose that $\mu(Z) > 0$. Then $\mu_\alpha(W) > 0$, and so $(\pi_\alpha^{\alpha+1})^{-1}(W)$ has non-empty interior. Hence

$$\operatorname{int}_K Z = \operatorname{int}_K(\pi_{\alpha+1}^{-1}((\pi_\alpha^{\alpha+1})^{-1}(W))) \neq \emptyset,$$

and so $\mu(Z) = 0$ whenever $Z \in \mathbf{Z}(K)$ and $\operatorname{int}_K Z = \emptyset$, i.e., μ is σ-normal by Theorem 4.7.4(ii). Since μ is strictly positive, K satisfies CCC, as is generally the case for the support of any $\mu \in M(K)$. By Proposition 4.7.7, $\mu \in N(K)$.

This completes the proof of the theorem. □

It can be shown, using the remark after Theorem 4.1.16, that $w(K) = \mathfrak{c}$, where K is the space of the above proof.

We have earlier defined a 'normal measure' on a Boolean algebra; see Definition 1.7.12. One might guess that a normal measure on a compact space K would give a normal measure on the Boolean algebra $\mathfrak{B}_K$. However this is not correct. Indeed, suppose that there exists $\mu \in N_c(K)^+$ with $\|\mu\| = 1$, and take the net (U_α) in $\mathfrak{B}_K$ consisting of the complements of the finite subsets of K, so that $U_\alpha \searrow 0$ in $\mathfrak{B}_K$, but $\mu(U_\alpha) = 1$ for each α, and so $\lim_{\alpha\in A} \mu(U_\alpha) \neq 0$. However, we do have the following result involving the Boolean algebra of regular–open sets, as defined in Example 1.7.16.

Theorem 4.7.27. *Let K be a non-empty, compact space. Then the map*

$$R : \mu \mapsto \mu \mid \mathfrak{R}_K\,, \quad N(K) \to N(\mathfrak{R}_K)\,,$$

is a Riesz isomorphism

Proof. Take $\mu \in N(K)$. Then it is clear that $R\mu$ is a measure on the Boolean algebra $\mathfrak{R}_K$ in the sense of Definition 1.7.12.

We first *claim* that $R\mu \in N(\mathfrak{R}_K)$. For this, it suffices to suppose that $\mu \in N(K)^+$. Take a net (U_α) with $U_\alpha \searrow \emptyset$ in $\mathfrak{R}_K$, and consider the set

$$\Gamma = \bigcup_\alpha \{f \in C(K) : \chi_{U_\alpha} \leq f\},$$

regarded as a downward-directed net in $C(K)^+$. Take $g \in C(K)^+$ with $g \leq f \ (f \in \Gamma)$; we shall show that $g = 0$. Indeed, assume towards a contradiction that $g \neq 0$. Then there is a non-empty, open set V in K with $g(x) > 0 \ (x \in V)$. Assume that α is such that $V \not\subset U_\alpha$. Then $V \not\subset \overline{U_\alpha}$ because U_α is regular–open, and so there exists $x \in V$ and $f \in C(K)$ with $f(x) = 0$ and $\chi_{U_\alpha} \leq f$, using the fact that K is compact. Thus $f \in \Gamma$, and hence $g(x) = 0$, a contradiction. This shows that $V \subset \bigcap U_\alpha$, a contradiction of the fact that $U_\alpha \searrow \emptyset$. Hence $g = 0$, and so $\inf \Gamma = 0$.

Since $\mu \in N(K)^+$, we see that $\inf\{\mu(f) : f \in \Gamma\} = 0$. However, for each $f \in \Gamma$, there exists α with $\chi_{U_\alpha} \leq f$, and so $\inf_\alpha \mu(U_\alpha) = 0$. We have shown that $R\mu$ satisfies the condition given in Definition 1.7.12 for it to be a normal measure on $\mathfrak{R}_K$, and so $R\mu \in N(\mathfrak{R}_K)^+$, giving the claim.

It is clear that $R : N(K) \to N(\mathfrak{R}_K)$ is a Riesz homomorphism.

We now *claim* that R is injective. Indeed, suppose that $\mu \in N_{\mathbb{R}}(K)$ with $R\mu = 0$. Then $R(|\mu|) = |R\mu| = 0$, and so $|\mu|(K) = R(|\mu|)(K) = 0$. Thus $\mu = 0$, and so R is injective, as claimed.

We finally *claim* that R is surjective. Indeed, take $\nu \in N(\mathfrak{R}_K)^+$, and define $\widehat{\mu}(B) = \nu(V_B) \ (B \in \mathfrak{B}_K)$, where V_B is the unique regular–open subset of K with $B \equiv V_B$.

We *claim* that $\widehat{\mu}$ is a measure on K. First, note that, for disjoint sets $B, C \in \mathfrak{B}_K$, we have $V_B \cap V_C \equiv B \cap C = \emptyset$, and so $\widehat{\mu}(B \cup C) = \widehat{\mu}(B) + \widehat{\mu}(C)$. Now suppose that (B_n) is an increasing sequence in $\mathfrak{B}_K$ with union $B \in \mathfrak{B}_K$. Then

$$B\Delta\left(\bigcup\{V_{B_n} : n \in \mathbb{N}\}\right) \subset \bigcup\{B_n \Delta V_{B_n} : n \in \mathbb{N}\}$$

is meagre. Set $U = \bigvee\{V_{B_n} : n \in \mathbb{N}\}$ in $\mathfrak{R}_K$, so that $U \Delta B$ is meagre and $U = V_B$. Then $\widehat{\mu}(B) = \nu(V_B) = \lim_{n\to\infty} \nu(V_{B_n})$ because ν is normal, and so $\widehat{\mu}(B) = \lim_{n\to\infty} \widehat{\mu}(B_n)$. This shows that $\widehat{\mu}$ is σ-additive. Thus $\widehat{\mu} \in M(K)$, and $\widehat{\mu}(B) \geq 0$ $(B \in \mathfrak{B}_K)$. (Note that it is not immediately obvious that $\widehat{\mu}$ is regular, but $\widehat{\mu}$ does define a continuous linear functional on $C(K)$.) By the Riesz representation theorem, there exists $\mu \in M(K)^+$ with

$$\int_K f \, \mathrm{d}\mu = \langle f, \widehat{\mu} \rangle \quad (f \in C(K)).$$

Let L be a non-empty, closed subspace of K. The family $\mathscr{U}$ of sets in $\mathfrak{R}_K$ that contain L is a net with infimum $\operatorname{int} L$ in $\mathfrak{R}_K$, and so $\{\nu(U) : U \in \mathscr{U}\}$ is a net in $\mathbb{R}$ with infimum $\nu(\operatorname{int} L)$. For each $U \in \mathscr{U}$, there exists $f_U \in C(K)$ with $\chi_L \leq f_U \leq \chi_U$, and then

$$\mu(L) \leq \int_K f_U \, \mathrm{d}\mu = \langle f, \widehat{\mu} \rangle \leq \widehat{\mu}(U) = \nu(U).$$

Thus $\mu(L) \leq \nu(\operatorname{int} L)$.

Take $U \in \mathfrak{R}_K$. By the previous remark, we have $\mu(U) = \mu(\operatorname{int} \overline{U}) \leq \nu(U)$, and hence $\mu(\operatorname{int}(K \setminus U)) \leq \nu(\operatorname{int}(K \setminus U))$, i.e., $\mu(U') \leq \nu(U')$, which implies that $\mu(U) \geq \nu(U)$. It follows that $\mu(U) = \nu(U)$.

For each $B \in \mathfrak{B}_K$, the set $B \Delta V_B$ is meagre, and so $\mu(B) = \mu(V_B) = \nu(V_B) = \widehat{\mu}(B)$. Thus $\mu = \widehat{\mu}$. Clearly $R\mu = \nu$ and so R is a surjection.

We conclude that $R : N(K) \to N(\mathfrak{R}_K)$ is a Riesz isomorphism. □

Corollary 4.7.28. *Let K and L be two compact spaces such that $\mathfrak{R}_K$ and $\mathfrak{R}_L$ are isomorphic as Boolean algebras. Then $N(K)$ and $N(L)$ are Banach-lattice isometric.*

Proof. Let $\rho : \mathfrak{R}_K \to \mathfrak{R}_L$ be an isomorphism, and then define

$$\widehat{\rho}(\mu)(V) = \mu(\rho^{-1}(V)) \quad (\mu \in N(\mathfrak{R}_K), V \in \mathfrak{R}_L),$$

so that $\widehat{\rho} : N(\mathfrak{R}_K) \to N(\mathfrak{R}_L)$ is the induced Riesz isomorphism. Next, let

$$R_K : N(K) \to N(\mathfrak{R}_K) \quad \text{and} \quad R_L : N(L) \to N(\mathfrak{R}_L)$$

be the Riesz isomorphisms given by the theorem. Set

$$T = R_L^{-1} \circ \widehat{\rho} \circ R_K : N(K) \to N(L).$$

Then T is a Riesz isomorphism. Further, $\|T\mu\| = |T\mu|(L) = |\mu|(K)$ $(\mu \in N(K))$ because $\rho^{-1}(L) = K$. By Proposition 2.3.5, there is a Banach-lattice isometry from $N(K)$ onto $N(L)$. □

We recall from Example 1.7.16 that $\mathfrak{R}_K$ and $\mathfrak{R}_L$ are isomorphic as Boolean algebras if and only if the Gleason covers G_K and G_L are homeomorphic. Thus Corollary 4.7.28 also follows easily from Corollary 4.7.19.