

**Lectures on
Disintegration of Measures**

**By
L. Schwartz**

**Tata Institute of Fundamental Research
Bombay
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Preface

THE CONTENTS OF these Notes have been given as a one month course of lectures in the Tata Institute of Fundamental Research in March 1974. The present reduction by Mr. S. Ramaswamy gives under a very short form the main results of my paper “Surmartingales régulières à valeurs mesures et désintégrations régulières d’une mesure” which appeared in the Journal d’Analyse Mathématique, Vol XXVI, 1973. For a first reading these Lecture Notes are better than the complete paper which however contains more results, in more general situations. Nothing is said here about stopping times. On the other hand, the integral representation with extremal elements of §3 in Chapter VII here, had not been published before.

L. Schwartz

Note

The material in these Notes has been divided into two parts. In part I, disintegration of a measure with respect to a single σ -algebra has been considered rather extensively and in part II, measure valued supermartingales and regular disintegration of a measure with respect to an increasing right continuous family of σ -algebras have been considered. The definition, remarks, lemmata, propositions, theorems and corollaries have been numbered in the same serial order. To each definition (resp. remark, lemma, proposition, theorem, corollary) there corresponds a triplet (a, b, c) where 'a' stands for the chapter and 'b' the section in which the definition (resp. remark, lemma, proposition, theorem, corollary) occurs and 'c' denotes its serial number. References to the bibliography have been indicated in square brackets.

The inspiring lectures of Professor L. Schwartz and the many discussions I had with him, have made the task of writing these Notes easier.

S. Ramaswamy

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Part I

Disintegration of a measure with respect to a single σ -Algebra

Chapter 1

Conditional expectations and disintegrations

1 Notations

Throughout these Notes the following notations will be followed. \mathbb{R} will stand for the set of all real numbers, $\bar{\mathbb{R}}$, for the set of all extended real numbers, i.e. \mathbb{R} together with the ideal points $-\infty$ and $+\infty$, \mathbb{N} for the set of all natural numbers, \mathbb{Z} for the set of all integers and \mathbb{Q} for the set of all rational numbers. \mathbb{R}^+ (resp $\bar{\mathbb{R}}^+$) will stand for the positive elements of \mathbb{R} (resp $\bar{\mathbb{R}}$).

Let X be a non-void set. Let A be a subset of X . Then $\complement A$ will stand for the complementary set of A , i.e. $\complement A = \{x \in X \mid x \notin A\}$.

Let X be a non-void set and \mathfrak{X} a σ -algebra of subsets of X . Then the pair (X, \mathfrak{X}) is called a *measurable space*. If f is a function on a measurable space (X, \mathfrak{X}) , with values in a topological space Y , $Y \neq \emptyset$, we say f *belongs to* \mathfrak{X} and write $f \in \mathfrak{X}$ if f is measurable with respect to \mathfrak{X} . (For measurability concepts, we always consider on topological spaces only their *Borel σ -algebra*, i.e., the σ -algebra generated by all the open sets).

By a *measure space* (X, \mathfrak{X}, μ) we always mean a measurable space (X, \mathfrak{X}) and a positive, $\neq 0$ measure μ on \mathfrak{X} .

Let (X, \mathfrak{X}, μ) be a measure space. Let A be a subset of X . We say μ

is *carried by* A if $\mu(\bar{C}A) = 0$ in case $A \in \mathfrak{X}$ and in case $A \notin \mathfrak{X}$, if $\mu(B)$ for every $B \in \mathfrak{X}$, $B \subset \bar{C}A$ is zero. $A \subset X$ is said to be a μ -null set if there exists a $B \in \mathfrak{X}$ with $\mu(B) = 0$ and $A \subset B$. \mathcal{N}_μ will stand for the class of all μ -null sets of X . $\hat{\mathfrak{X}}_\mu$ will stand for the σ -algebra generated by \mathfrak{X} and \mathcal{N}_μ .

- 2 $\hat{\mathfrak{X}}_\mu$ is called the *completion* of \mathfrak{X} with respect to μ . If $\mathfrak{X} = \hat{\mathfrak{X}}_\mu$, we say \mathfrak{X} is *complete* with respect to μ . If f is a function on X with values in a topological space Y , $Y \neq \emptyset$ and if $f \in \hat{\mathfrak{X}}_\mu$, we say f is μ -*measurable*. If \mathcal{Y} is any σ -algebra on X , we denote by $\mathcal{Y} \vee \mathcal{N}_\mu$, the σ -algebra generated by \mathcal{Y} and \mathcal{N}_μ . Thus, $\hat{\mathfrak{X}}_\mu = \mathfrak{X} \vee \mathcal{N}_\mu$.

The symbol $\forall_{\mu\text{-a.e.}}$ will stand for 'for μ -almost all x '.

If h is any non-negative function $\in \hat{\mathfrak{X}}_\mu$, h, μ will stand for the measure on $\hat{\mathfrak{X}}_\mu$ given by $h \cdot \mu(B) = \int_B h(w) d\mu(w)$ for all $B \in \hat{\mathfrak{X}}_\mu$.

If E is a Banach space over the real numbers, $L^1(X; \mathfrak{X}; \mu; E)$ will stand for the Banach space of all μ -equivalence classes of functions on X with values in E which belong to $\hat{\mathfrak{X}}_\mu$ and which are μ -integrable.

$L^1(X; \mathfrak{X}; \mu)$ will stand for the Banach space of all μ -equivalence classes of functions with values in \mathbb{R} which belong to $\hat{\mathfrak{X}}_\mu$ and which are μ -integrable.

Let $\mathcal{A}(X; \mathfrak{X}; \mu)$ denote the set of all extended real valued functions on X which belong to $\hat{\mathfrak{X}}_\mu$ and let $\mathcal{A}^+(X; \mathfrak{X}; \mu)$ denote the set of all non-negative elements of $\mathcal{A}(X; \mathfrak{X}; \mu)$.

If $f \in \mathcal{A}^+(X; \mathfrak{X}; \mu)$ or if $f \in L^1(X; \mathfrak{X}; \mu)$ or if $f \in L^1(X; \mathfrak{X}; \mu; E)$, then $\mu(f)$, $\int f(w) d\mu(w)$, $\int f(w) \mu(dw)$, $\int f d\mu$ will all denote the integral of f with respect to μ , over X .

2 Basic definitions in the theory of integration for Banach space valued functions

The theory of integration for Banach space valued functions on a measure space is assumed here. However, by way of recalling, we give below a few basic definitions.

- 3 Let $(\Omega, \mathcal{O}, \lambda)$ be a measure space. Let E be a Banach space over the

real numbers. If \mathcal{S} is any σ -algebra on Ω , a function f on Ω with values in E is said to be a *step function* belonging to \mathcal{S} , if there exist finitely many sets $(A_i)_{i=1,2,\dots,n}$, $A_i \in \mathcal{S}$, $i = 1, \dots, n$, $A_i \cap A_j = \emptyset$ if $i \neq j$ and finitely many points $(x_i)_{i=1,\dots,n}$, $x_i \in E \quad \forall i = 1, \dots, n$ such that $\forall w \in \Omega$, $f(w) = \sum_{i=1}^n \chi_{A_i}(w)x_i$. A function f on Ω with values in E is said to be *strongly measurable* if there exists a sequence $(f_n)_{n \in \mathbb{N}}$ of step functions, $f_n \in \hat{\mathcal{O}}_\lambda \forall n \in \mathbb{N}$, such that $\forall_\lambda w$, $f_n(w) \rightarrow f(w)$ in E , as $n \rightarrow \infty$. Note that a strongly measurable function belongs to $\hat{\mathcal{O}}_\lambda$. A function f on Ω with values in E is said to be λ -*integrable* or *integrable in the sense of Bochner* if f is strongly measurable and if $\int |f|(w)d\lambda(w) < \infty$, where $|f|$ is the real valued function on Ω assigning to each $w \in \Omega$, the norm of $f(w)$ in E . One can prove that if \mathcal{S} is a σ -algebra contained in $\hat{\mathcal{O}}_\lambda$ and if $f \in \mathcal{S}$ is λ -integrable, then one can find a sequence $(f_n)_{n \in \mathbb{N}}$ of step functions belonging to \mathcal{S} and a real valued non-negative λ -integrable function g belonging to \mathcal{S} such that

$$\begin{aligned} \forall_\lambda w, f_n(w) &\rightarrow f(w) \text{ in } E \text{ as } n \rightarrow \infty \text{ and} \\ \forall n, \forall_\lambda w, |f_n|(w) &\leq g(w). \end{aligned}$$

For further properties of Bochner integrals, the reader is referred to Hille and Phillips [1].

3 Conditional Expectations and Disintegrations; Basic definitions

Let $(\Omega, \mathcal{O}, \lambda)$ be a measure space. Let \mathcal{C} be a σ -algebra contained in $\hat{\mathcal{O}}_\lambda$.

Definition 1. Let f be a λ -integrable function on Ω with values in a Banach space E over the reals (resp. f extended real valued, $f \geq 0$ and $f \in \hat{\mathcal{O}}_\lambda$). A function $f^\mathcal{C}$ on Ω with values in E (resp. extended reals) is said to be a *conditional expectation of f with respect to \mathcal{C}* if **4**

(i) $f^\mathcal{C} \in \mathcal{C}$ and is λ -integrable (resp. (i) $f^\mathcal{C} \in \mathcal{C}$ and is ≥ 0) and (ii) $\forall A \in \mathcal{C}$, $\int_A f^\mathcal{C}(w)d\lambda(w) = \int_A f(w)d\lambda(w)$.

Here, $\int_A f(w)d\lambda(w)$ (resp. $\int_A f^{\mathcal{C}}(w)d\lambda(w)$) stands for the integral of $\chi_A \cdot f$ (resp. $\chi_A \cdot f^{\mathcal{C}}$) with respect to λ . These exist since f and $f^{\mathcal{C}}$ are λ -integrable (resp. f and $f^{\mathcal{C}}$ are ≥ 0).

Definition 2. A family $(\lambda_w^{\mathcal{C}})_{w \in \Omega}$ of positive measures on \mathcal{O} , indexed by Ω is called a system of conditional probabilities with respect to \mathcal{C} or a disintegration of λ with respect to \mathcal{C} if it has the following properties, namely

(i) $\forall B \in \mathcal{O}$, the function $w \rightarrow \lambda_w^{\mathcal{C}}(B)$ belongs to \mathcal{C}

and (ii) $\forall B \in \mathcal{O}$, the function $w \rightarrow \lambda_w^{\mathcal{C}}(B)$ is a conditional expectation of χ_B with respect to \mathcal{C} .

We remark that (i) implies that for every function f on Ω , $f \geq 0$, $f \in \mathcal{O}$, the function $w \rightarrow \lambda_w^{\mathcal{C}}(f)$ belongs to \mathcal{C} and that (ii) implies that \forall function f on Ω , $f \geq 0$, $f \in \mathcal{O}$, the function $w \rightarrow \lambda_w^{\mathcal{C}}(f)$ belongs to \mathcal{C} and that (ii) implies that \forall function f on Ω , $f \geq 0$, $f \in \mathcal{O}$, the function $w \rightarrow \lambda_w^{\mathcal{C}}(f)$ is a conditional expectation of f with respect to \mathcal{C} . From (ii), taking $B = \Omega$, we see that $\forall \lambda w$, $\lambda_w^{\mathcal{C}}$ is a probability measure.

Note that the existence of a disintegration of λ with respect to \mathcal{C} implies immediately the existence of conditional expectations with respect to \mathcal{C} for non-negative functions belonging to \mathcal{O} . We shall see below that if λ restricted to \mathcal{C} is σ -finite, conditional expectation with respect to \mathcal{C} for any non-negative function belonging to $\hat{\mathcal{O}}_{\lambda}$ and for any λ -integrable Banach space valued function exists. But a disintegration of λ need not always exist without any further assumptions about Ω and \mathcal{O} as can be seen by an example due to J. Dieudonné [1].

4 Illustrations and motivations

- 5 Let $(\Omega, \mathcal{O}, \lambda)$ be a measure space. Before we proceed to prove the existence theorems of conditional expectations, we shall see the above notions when \mathcal{C} is given by a partition of Ω .

Let $A \in \hat{\mathcal{O}}_{\lambda}$ be such that $0 < \lambda(A) < \infty$. We call the measure $\frac{\chi_A \cdot \lambda}{\lambda(A)}$

on \mathcal{O} , the *conditional probability* of λ given A . We call $\frac{\int_A f d\lambda}{\lambda(A)}$, the *conditional expectation* of f given A , where f is either a Banach space valued λ -integrable function or a non-negative function belonging to $\hat{\mathcal{O}}_\lambda$. Note that the conditional probability is always a probability measure whether λ is or not.

Suppose also that $0 < \lambda(\mathbb{C}A) < \infty$. (This will happen only if λ is a finite measure and when $0 < \lambda(A) < \lambda(\Omega)$). Then, consider the σ -algebra $\mathcal{C} = (\emptyset, A, \mathbb{C}A, \Omega)$.

Let E be a Banach space over the real numbers and let f be a λ -integrable function on Ω with values in E (resp. $f \geq 0$, $f \in \hat{\mathcal{O}}_\lambda$). Define

$$f^{\mathcal{C}}(w) = \begin{cases} \int f d\lambda \\ \frac{\int_A f d\lambda}{\lambda(A)}, & \text{if } w \in A \\ \int f d\lambda \\ \frac{\int_{\mathbb{C}A} f d\lambda}{\lambda(\mathbb{C}A)}, & \text{if } w \in \mathbb{C}A \end{cases}$$

Then $f^{\mathcal{C}}$ is with values in E , $f^{\mathcal{C}} \in \mathcal{C}$ and is λ -integrable (resp. $f^{\mathcal{C}}$ is ≥ 0 , $f^{\mathcal{C}} \in \mathcal{C}$).

We have

- (i) $\int_A f^{\mathcal{C}}(w) d\lambda(w) = \int_A f(w) d\lambda(w)$ and
(ii) $\int_A f^{\mathcal{C}}(w) d\lambda(w) = \int_{\mathbb{C}A} f(w) d\lambda(w)$.

Thus, $f^{\mathcal{C}}$ is a conditional expectation of f with respect to \mathcal{C} .

Consider the family $(\lambda_w^{\mathcal{C}})_{w \in \Omega}$ of measures on \mathcal{O} defined as

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$$\lambda_w^{\mathcal{C}} = \begin{cases} \frac{\chi_A \cdot \lambda}{\lambda(A)}, & \text{if } w \in A \\ \frac{\chi_{\mathbb{C}A} \cdot \lambda}{\lambda(\mathbb{C}A)}, & \text{if } w \in \mathbb{C}A. \end{cases}$$

This family $(\lambda_w^{\mathcal{C}})_{w \in \Omega}$ satisfies the conditions (i) and (ii) of definition (1, § 3, 2) and hence is a disintegration of λ with respect to \mathcal{C} .

Thus, we see that the conditional expectation of any Banach space valued λ -integrable function, that of any non-negative function belonging to $\hat{\mathcal{O}}_\lambda$ and a disintegration of λ with respect to \mathcal{C} , always exist when \mathcal{C} is of the form $(\emptyset, A, \mathbb{C}A, \Omega)$ with $0 < \lambda(A) < \infty$ and $0 < \lambda(\mathbb{C}A) < \infty$.

Next, let us consider the following situation.

Suppose $\Omega = \bigcup_{n \in \mathbb{N}} A_n$, where $\forall n, A_n \in \hat{\mathcal{O}}_\lambda, A_n \cap A_m = \emptyset$, if $n \neq m$ and $0 < \lambda(A_n) < \infty \forall n \in \mathbb{N}$. We call such a sequence $(A_n)_{n \in \mathbb{N}}$ of sets, a *partition* of Ω . Let I be any subset of \mathbb{N} . Then, the collection of all sets of the form $\bigcup_{i \in I} A_i$, as I varies over all the subsets of \mathbb{N} is a σ -algebra. Let us denote this σ -algebra by \mathcal{C} . We say then that \mathcal{C} is given by the partition $(A_n)_{n \in \mathbb{N}}$.

If f is a λ -integrable function with values in E or if f is ≥ 0 and belongs to $\hat{\mathcal{O}}_\lambda$, consider the function $f^\mathcal{C}$ defined as

$$f^\mathcal{C}(w) = \frac{\int_{A_n} f d\lambda}{\lambda(A_n)}, \text{ if } w \in A_n.$$

Then $f^\mathcal{C}$ is easily seen to be a conditional expectation of f with respect to \mathcal{C} .

Consider the family $(\lambda_w^\mathcal{C})_{w \in \Omega}$ of measures on \mathcal{O} defined as

$$\lambda_w^\mathcal{C} = \frac{\chi_{A_n} \cdot \lambda}{\lambda(A_n)} \text{ if } w \in A_n.$$

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Then the family $(\lambda_w^\mathcal{C})_{w \in \Omega}$ of measures satisfies the condition (i) and (ii) of definition (1, § 3, 2) and hence is a disintegration of λ with respect to \mathcal{C} .

Thus, we see that when \mathcal{C} is given by a partition $(A_n)_{n \in \mathbb{N}}$ with $0 < \lambda(A_n) < \infty$, conditional expectation of any Banach space valued λ -integrable function, that of any non-negative function belonging to $\hat{\mathcal{O}}_\lambda$ and a disintegration of λ with respect to \mathcal{C} , always exist.

5 Existence and uniqueness theorems of conditional expectations for extended real valued functions; A few properties

Let $(\Omega, \mathcal{O}, \lambda)$ be a measure space. Let \mathcal{C} be a σ -algebra contained in $\hat{\mathcal{O}}_\lambda$. Let further, λ restricted to \mathcal{C} be σ -finite.

Proposition 3. *Let $f \in L^1(\Omega; \mathcal{O}; \lambda)$ (resp. $f \in \mathcal{A}^+(\Omega; \mathcal{O}; \lambda)$). Then a conditional expectation of f with respect of \mathcal{C} exists. Moreover, it is unique in the sense that if g_1 and g_2 are two conditional expectations of f with respect to \mathcal{C} , then $\forall \lambda w, g_1(w) = g_2(w)$.*

Proof. The proof of this proposition follows just from a straight forward application of the Radon-Nikodym theorem. More precisely, for $A \in \mathcal{C}$, define $\nu(A) = \int_A f(w) d\lambda(w)$. Then ν is a finite signed measure (resp. a positive measure) on \mathcal{C} and is absolutely continuous with respect to λ which is σ -finite on \mathcal{C} . Hence, by the Radon-Nikodym theorem, there exists a function $g \in \mathcal{C}$ which is λ -integrable (resp. $g \in \mathcal{C}$ and is ≥ 0) and which is unique upto a set of measure zero, such that

$$\nu(A) = \int_A f(w) d\lambda(w) = \int_A g(w) d\lambda(w) \quad \forall A \in \mathcal{C}.$$

□

Note that for any given $f \in L^1(\Omega; \mathcal{O}; \lambda)$ (resp. $f \in \mathcal{A}^+(\Omega; \mathcal{O}; \lambda)$), we get a class of functions as conditional expectations of f with respect to \mathcal{C} , any two functions in the class differing by a \mathcal{C} -set of measure zero at most. Also note that if f_1 and f_2 are any two functions belonging to $\mathcal{A}^+(\Omega; \mathcal{O}; \lambda)$ and are equal almost everywhere, then also we have $\forall \lambda w, f_1^{\mathcal{C}}(w) = f_2^{\mathcal{C}}(w)$. Thus, we have a map $u_{\mathcal{O}, \mathcal{C}}$ (resp. $v_{\mathcal{O}, \mathcal{C}}$) from $L^1(\Omega; \mathcal{O}; \lambda)$ to $L^1(\Omega; \mathcal{O}; \lambda)$ (resp. from $\mathcal{A}^+(\Omega; \mathcal{O}; \lambda)$ to $\tilde{\mathcal{A}}_{\mathcal{C}}^+$ where $\tilde{\mathcal{A}}_{\mathcal{C}}^+$ stands for the set of all λ -equivalence classes of non-negative functions which belong to \mathcal{C}). Here, we are making an abuse of notation by denoting by the same symbol f , a function as well as the class to which it belongs.

The map $u_{\mathcal{O}, \mathcal{C}}$ is a continuous linear map from the Banach space $L^1(\Omega; \mathcal{O}; \lambda)$ to $L^1(\Omega; \mathcal{C}; \lambda)$ with $\|u_{\mathcal{O}, \mathcal{C}}\| = 1$ where $\|u_{\mathcal{O}, \mathcal{C}}\| = \sup_{f \neq 0} \frac{\int |f^{\mathcal{C}}| d\lambda}{\int |f| d\lambda}$.

We have the following properties of the maps $u_{\mathcal{O}, \mathcal{C}}$ and $v_{\mathcal{O}, \mathcal{C}}$.

(i) $u_{\mathcal{O}, \mathcal{C}}(f^{\mathcal{C}}) = f^{\mathcal{C}} \forall f \in L^1(\Omega; \mathcal{O}; \lambda)$ i.e. $u_{\mathcal{O}, \mathcal{C}}$ is a projection onto the subspace $L^1(\Omega; \mathcal{C}; \lambda)$ of $L^1(\Omega; \mathcal{O}; \lambda)$

$$(v_{\mathcal{O}, \mathcal{C}}(f^{\mathcal{C}})) = f^{\mathcal{C}} \forall f \in \mathcal{A}^+(\Omega; \mathcal{O}; \lambda)$$

(ii) $|u_{\mathcal{O}, \mathcal{C}}(f)| \leq u_{\mathcal{O}, \mathcal{C}}(|f|)$ in the sense that $\forall f \in L^1(\Omega; \mathcal{O}; \lambda)$, $\forall_{\lambda} w$, $|u_{\mathcal{O}, \mathcal{C}}(f)|(w) \leq u_{\mathcal{O}, \mathcal{C}}(|f|)(w)$.

(iii) $f \geq 0 \Rightarrow u_{\mathcal{O}, \mathcal{C}}(f) \geq 0$ in the sense that $\forall f \in L^1(\Omega; \mathcal{O}; \lambda)$ which is such that $\forall_{\lambda} w$, $f(w) \geq 0$, we have $\forall_{\lambda} w$, $u_{\mathcal{O}, \mathcal{C}}(f)(w) \geq 0$.

(iv) If g is non-negative and belongs to \mathcal{C} and $f \in L^1(\Omega; \mathcal{O}; \lambda)$ and if gf is λ -integrable or if g is λ -integrable and belongs to \mathcal{C} such that gf is λ -integrable, we have

$$u_{\mathcal{O}, \mathcal{C}}(fg) = g \cdot u_{\mathcal{O}, \mathcal{C}}(f).$$

9 (If g is non-negative and belongs to \mathcal{C}) and if $f \in \mathcal{A}^+(\Omega; \mathcal{O}; \lambda)$, then

$$(\forall_{\lambda} w, v_{\mathcal{O}, \mathcal{C}}(fg)(w) = g(w) \cdot v_{\mathcal{O}, \mathcal{C}}(f)(w)).$$

(v) If \mathcal{S} is any σ -algebra contained in $\hat{\mathcal{C}}_{\lambda}$,

$$u_{\mathcal{C}, \mathcal{S}} \circ u_{\mathcal{O}, \mathcal{C}} = u_{\mathcal{O}, \mathcal{S}}$$

i.e. the operation of conditional expectation is transitive

$$(v_{\mathcal{C}, \mathcal{S}} \circ v_{\mathcal{O}, \mathcal{C}} = v_{\mathcal{O}, \mathcal{S}})$$

(vi) $u_{\mathcal{O}, \mathcal{O}} = \text{Identity}$.

$$(\forall f \in \mathcal{A}^+(\Omega; \mathcal{O}; \lambda), \quad \forall_{\lambda} w, v_{\mathcal{O}, \mathcal{O}}(f)(w) = f(w))$$

(vii) $u_{\mathcal{O}, (\Omega, \emptyset)}(f) = \frac{1}{\lambda(\Omega)} \int f d\lambda$ if $\lambda(\Omega)$ is finite

$$(\forall f \in \mathcal{A}^+(\Omega; \mathcal{O}; \lambda), \quad \forall_{\lambda} w, v_{\mathcal{O}, (\Omega, \emptyset)}(f)(w) = \frac{1}{\lambda(\Omega)} \int f d\lambda$$

if $\lambda(\Omega) < \infty$)

6 Existence and Uniqueness theorems of conditional expectation for Banach space valued integrable functions

Let $(\Omega, \mathcal{O}, \lambda)$ be a measure space, and let \mathcal{C} be a σ -algebra contained in $\hat{\mathcal{O}}_\lambda$. Let λ restricted to \mathcal{C} be σ -finite. Let E be a Banach space over the real numbers.

Now, we are going to prove the existence and uniqueness theorems of conditional expectations for any $f \in L^1(\Omega; \mathcal{O}; \lambda; E)$. To prove the existence theorem, we have to adopt essentially a different method than the one in § 5 for extended real numbers as the Radon-Nikodym theorem in general is not valid for Banach spaces. By the Radon-Nikodym theorem for Banach spaces we mean the following theorem: 10

Let (X, \mathfrak{X}, μ) be a measure space. Let E be a Banach space over the real numbers. Let ν be a measure on \mathfrak{X} with values in E and let ν be absolutely continuous with respect to μ . Then, there exists a μ -integrable function g on \mathfrak{X} with values in E such that

$$\forall A \in \mathfrak{X}, \quad \nu(A) = \int_A g d\mu.$$

Theorem 4. *Let $f \in L^1(\Omega; \mathcal{O}; \lambda; E)$. Then,*

- (i) Existence: *A conditional expectation of f with respect to \mathcal{C} exists.*
- (ii) Uniqueness: *If g_1 and g_2 are two conditional expectations of f with respect to \mathcal{C} , then $\forall \lambda w, g_1(w) = g_2(w)$.*

Proof. (i) *Existence.* Let $L^1(\Omega; \mathcal{O}, \lambda) \otimes_{\mathbb{R}} E$ be the algebraic tensor product of $L^1(\Omega; \mathcal{O}; \lambda)$ and E over the real numbers. There is an injective linear map from $L^1(\Omega; \mathcal{O}; \lambda) \otimes_{\mathbb{R}} E$ to $L^1(\Omega; \mathcal{O}; \lambda; E)$ which takes an element $f \otimes x$ of $L^1(\Omega; \mathcal{O}; \lambda) \otimes_{\mathbb{R}} E$ to $f.x$ of $L^1(\Omega; \mathcal{O}; \lambda; E)$. Hence, we can consider $L^1(\Omega; \mathcal{O}; \lambda) \otimes_{\mathbb{R}} E$ as a subspace of $L^1(\Omega; \mathcal{O}; \lambda; E)$. A theorem of Grothendieck says that

$L^1(\Omega; \mathcal{O}; \lambda; E)$ is the completion of $L^1(\Omega; \mathcal{O}; \lambda) \otimes_{\mathbb{R}} E$ for the ‘ π -topology’ on $L^1(\Omega; \mathcal{O}; \lambda) \otimes_{\mathbb{R}} E$.

The linear map v from $L^1(\Omega; \mathcal{O}; \lambda) \otimes_{\mathbb{R}} E$ to $L^1(\Omega; \mathcal{C}; \lambda) \otimes_{\mathbb{R}} E$ which takes an element $f \otimes x$ of $L^1(\Omega; \mathcal{O}; \lambda) \otimes_{\mathbb{R}} E$ to $u_{\mathcal{O}, \mathcal{C}}(f) \otimes x$ of $L^1(\Omega; \mathcal{C}; \lambda) \otimes_{\mathbb{R}} E$ is a contraction mapping of these normed spaces under the respective ‘ π -topologies’ and hence extends to a unique continuous linear mapping of $L^1(\Omega; \mathcal{O}; \lambda; E)$ to $L^1(\Omega; \mathcal{C}; \lambda; E)$, which we again denote by v . Now, it is easy to see that $\forall f \in L^1(\Omega; \mathcal{O}; \lambda; E)$, $v(f)$ is a conditional expectation of f with respect to \mathcal{C} .

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(ii) *Uniqueness.* To prove uniqueness, it is sufficient to prove that if f is a λ -integrable function on Ω with values in E and belongs to \mathcal{C} and if $\int_A f d\lambda = 0 \forall A \in \mathcal{C}$, then $\forall \lambda w, f(w) = 0$.

So, let $f \in L^1(\Omega; \mathcal{C}; \lambda; E)$ with $\int_A f d\lambda = 0 \forall A \in \mathcal{C}$.

Let E' be the topological dual of E . If $x' \in E'$ and $x \in E$, let $\langle x', x \rangle$ denote the value of x' at x , $\forall x' \in E', \forall A \in \mathcal{C}$, we have $\int_A \langle x', f(w) \rangle d\lambda(w) = 0$, and therefore, $\forall x' \in E', \forall \lambda w, \langle x', f(w) \rangle = 0$.

□

Now, there exists a set $N_1 \in \mathcal{C}$ with $\lambda(N_1) = 0$, and a *separable* subspace F of E such that if $w \notin N_1, f(w) \in F$. This is because, f is the limit almost everywhere of step functions. Since F is separable, there exists a countable set $(x'_n)_{n \in \mathbb{N}}, x'_n \in F' \forall n \in \mathbb{N}$, such that

$$\forall x \in F, \|x\| = \sup_n |\langle x'_n, x \rangle|$$

(See Hille and Phillips [1], p.34, theorem 2.8.5). Therefore, if $w \notin N_1$,

$$|f|(w) = \sup_n |\langle x'_n, f(w) \rangle|$$

where $|f|(w)$ is the norm of the element $f(w)$.

Since $\forall x' \in E', \forall_\lambda w, \langle x', f(w) \rangle = 0$, we can find a set $N_2 \in \mathcal{C}$ with $\lambda(N_2) = 0$ such that if $w \notin N_2, \langle x'_n, f(w) \rangle = 0 \quad \forall n \in \mathbb{N}$.

Therefore, if $w \notin N_1 \cup N_2$,

$$|f|(w) = \sup_n |\langle x'_n, f(w) \rangle| = 0.$$

This shows that $\forall_\lambda w, f(w) = 0$.

7 Another way of proving the existence theorem of conditional expectations for Banach space valued integrable functions

The existence of conditional expectation for Banach space valued integrable functions can be proved in the following way also. 12

As before, let $(\Omega, \mathcal{O}, \lambda)$ be a measure space. Let \mathcal{C} be a σ -algebra contained in $\hat{\mathcal{O}}_\lambda$. Let λ restricted to \mathcal{C} be σ -finite. Let E be a Banach space over the real numbers. Let E' be the topological dual of E . If $x' \in E'$ and $x \in E, \langle x', x \rangle$ will stand for the value of x' at x . If f is a function on Ω with values in E and if ξ is a function on Ω with values in $E', \langle \xi, f \rangle$ will stand for the real valued function on Ω associating to each $w \in \Omega$, the real number $\langle \xi(w), f(w) \rangle$. If f and ξ are as above, $|f|$ (resp. $|\xi|$) will denote the function on Ω associating to each $w \in \Omega$, the norm in E of the element $f(w)$ (resp. the norm in E' of the element $\xi(w)$).

Let f be a function on Ω with values in E such that $f(\Omega)$ is contained in a finite dimensional subspace of E . We call such a function a *finite dimensional valued function*. Let F be a finite dimensional subspace of E containing $f(\Omega)$ and let e_1, e_2, \dots, e_n be a basis for F . Then there exist n real valued functions $\alpha_1, \alpha_2, \dots, \alpha_n$ on Ω such that $\forall w \in \Omega, f(w) = \sum_{i=1}^n \alpha_i(w)e_i$. If \mathcal{S} is any σ -algebra on Ω , it is clear that $f \in \mathcal{S}$ if and only if $\forall i = 1, 2, \dots, n, \alpha_i \in \mathcal{S}$ and that if $f \in \mathcal{S}$ and if $\mathcal{S} \subset \hat{\mathcal{O}}_\lambda$, then f is λ -integrable if and only if $\forall i = 1, \dots, n, \alpha_i$ is λ -integrable. If f is λ -integrable, then a conditional expectation $f^{\mathcal{C}}$ of f with respect to

\mathcal{C} is defined as $f^{\mathcal{C}}(w) = \sum_{i=1}^n \alpha_i^{\mathcal{C}}(w)e_i \forall w \in \Omega$, where $\forall i = 1, \dots, n$, $\alpha_i^{\mathcal{C}}$ is a conditional expectation of α_i with respect to \mathcal{C} . Note that $\alpha_i^{\mathcal{C}}$ exist $\forall i = 1, \dots, n$. It can be easily seen that this definition is independent of the choice of the basis of F and also in independent of the finite dimensional subspace of E containing $f(\Omega)$.

Remark 5. If f is a finite dimensional valued λ -integrable function on Ω with values in E and if ξ is any function on Ω with values in E' such that $\xi \in \mathcal{C}$ (on E' we always consider the Borel σ -algebra of the strong topology) and ξ is bounded in the sense that $\sup_{w \in \Omega} |\xi|(w)$ is a real number, then,

$$\forall_{\lambda} w, \langle \xi, f \rangle^{\mathcal{C}}(w) = \langle \xi, f^{\mathcal{C}} \rangle(w).$$

This is a consequence of the property (iv) of conditional expectations of extended real valued functions listed in § 5 of this chapter.

The alternative proof of the existence of conditional expectations of Banach space valued functions depends on the following theorem.

Theorem 6. Let f be a finite dimensional valued λ -integrable function on Ω with values in a Banach space E . Then,

$$\forall_{\lambda} w, |f^{\mathcal{C}}|(w) \leq |f|^{\mathcal{C}}(w).$$

To prove this theorem, we need the following lemma.

Lemma 7. Let \mathcal{S} be a σ -algebra on Ω .

- (i) Let f be a step function on Ω with values in E , belonging to \mathcal{S} . Then there exists a function $\xi : \Omega \rightarrow E'$, $\xi \in \mathcal{S}$, $|\xi| \leq 1$ such that $\forall w \in \Omega$, $\langle \xi(w), f(w) \rangle = |f|(w)$.
- (ii) If $\mathcal{S} \subset \hat{\mathcal{O}}_{\lambda}$ and if f is any λ -integrable function on Ω with values in E belonging to \mathcal{S} , then there exists a sequence $(\xi_n)_{n \in \mathbb{N}}$ of functions on Ω with values in E' , $\xi_n \in \mathcal{S}$ and $|\xi_n| \leq 1 \forall n \in \mathbb{N}$, such that

$$\forall_{\lambda} w, \lim_{n \rightarrow \infty} \langle \xi_n(w), f(w) \rangle = |f|(w).$$

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Proof. (i) Let f , a step function be of the form $\sum_{i=1}^n \chi_{A_i} x_i$, $A_i \in \mathcal{S}$ $\forall i = 1, \dots, n$, $A_i \cap A_j = \emptyset$ if $i \neq j$, $x_i \in E \forall i = 1, \dots, n$. By Hahn-Banach theorem, there exists $\forall i, i = 1, \dots, n$, an element $\xi_i \in E'$ such that $\langle \xi_i, x_i \rangle = \|x_i\|$ and $\|\xi_i\| \leq 1$. Let $\xi = \sum_{i=1}^n \chi_{A_i} \cdot x_i$. Then it is easily seen that ξ has all the required properties.

(ii) Let f be an arbitrary λ -integrable function belonging to \mathcal{S} . Then there exist a sequence $(f_n)_{n \in \mathbb{N}}$ of step functions belonging to \mathcal{S} such that

$$\forall_{\lambda} w, |f_n - f|(w) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

By (i) $\forall n, \exists \xi_n : \Omega \rightarrow E', \xi_n \in \mathcal{S}, |\xi_n| \leq 1$ such that $\langle \xi_n(w), f_n(w) \rangle = |f_n(w)| \forall w \in \Omega$.

$$\begin{aligned} & \left| \langle \xi_n(w), f(w) \rangle - |f|(w) \right| \\ &= \left| \langle \xi_n(w), f(w) - f_n(w) \rangle + \langle \xi_n(w), f_n(w) \rangle - |f|(w) \right| \\ &= \left| \langle \xi_n(w), f(w) - f_n(w) \rangle + |f_n|(w) - |f|(w) \right| \\ &\leq |f - f_n|(w) + |f_n - f|(w) \\ &= 2|f - f_n|(w). \end{aligned}$$

Hence, $\forall_{\lambda} w, \lim_{n \rightarrow \infty} \langle \xi_n(w), f(w) \rangle$ exists and is equal to $|f|(w)$. \square

Proof of the theorem 6. Applying the above lemma (1, § 7, 7) to $f^{\mathcal{C}}$, we see that since $f^{\mathcal{C}} \in \mathcal{C}$, there exists a sequence $(\xi_n)_{n \in \mathbb{N}}$ of functions on Ω with values in E' such that $\xi_n \in \mathcal{C} \forall n \in \mathbb{N}, |\xi_n| \leq 1 \forall n \in \mathbb{N}$ and $\forall_{\lambda} w, \lim_{n \rightarrow \infty} \langle \xi_n(w), f^{\mathcal{C}}(w) \rangle = |f^{\mathcal{C}}|(w)$.

By remark (1, § 7, 5),

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$$\forall n \in \mathbb{N}, \forall_{\lambda} w, \langle \xi_n(w), f^{\mathcal{C}}(w) \rangle = \langle \xi_n, f \rangle^{\mathcal{C}}(w).$$

Therefore, $\forall n \in \mathbb{N}, \forall \lambda w, |\langle \xi_n(w), f^{\mathcal{C}}(w) \rangle| = |\langle \xi_n, f \rangle^{\mathcal{C}}(w)|$ and $\forall n \in \mathbb{N}, \forall \lambda w, |\langle \xi_n, f \rangle^{\mathcal{C}}(w)| \leq |\langle \xi_n, f \rangle^{\mathcal{C}}(w)|$ by property (ii) of the conditional expectations of extended real valued functions, listed in § 5 of this chapter.

Now, $\forall n \in \mathbb{N}, |\langle \xi_n, f \rangle| \leq \xi_n \cdot |f| \leq |f|$ and hence, $\forall n \in \mathbb{N}, \forall \lambda w, |\langle \xi_n, f \rangle^{\mathcal{C}}(w)| \leq |f|^{\mathcal{C}}(w)$. Hence $\forall \lambda w, \forall n \in \mathbb{N}, |\langle \xi_n, f \rangle^{\mathcal{C}}(w)| \leq |f|^{\mathcal{C}}(w)$. Hence, $\forall \lambda w, \forall n \in \mathbb{N}, \left| \langle \xi_n(w), f^{\mathcal{C}}(w) \rangle \right| \leq |f|^{\mathcal{C}}(w)$. Hence, $\forall \lambda w, |f^{\mathcal{C}}(w)| \leq |f|^{\mathcal{C}}(w)$.

Remark 8. Actually, one can prove that if f is a finite dimensional valued λ -integrable function, belonging to a σ -algebra \mathcal{S} contained in $\hat{\mathcal{O}}_\lambda$, then there exists a function ξ on Ω with values in E' , $\xi \in \mathcal{S}$, $|\xi| \leq 1$ such that $\forall w \in \Omega, \langle \xi(w), f(w) \rangle = |f|(w)$. But this is very difficult. Note that once this is proved, the proof of theorem (1, § 7, 6) follows more easily.

Now, let us turn to the proof of the existence theorem of conditional expectations for Banach space valued λ -integrable functions.

Let m be the vector subspace of all finite dimensional valued functions belonging to $\hat{\mathcal{O}}_\lambda$ and λ -integrable. Then m is dense in $L^1(\Omega; \mathcal{O}; \lambda; E)$, as we can approximate any $f \in L^1(\Omega; \mathcal{O}; \lambda; E)$ by step functions. Consider the linear map $u_{\mathcal{O}, \mathcal{C}}$ from m to $L^1(\Omega; \mathcal{C}; \lambda; E)$ given by

$$u_{\mathcal{O}, \mathcal{C}}(f) = f^{\mathcal{C}}.$$

$$\int |u_{\mathcal{O}, \mathcal{C}}(f)|(w) d\lambda(w) = \int |f^{\mathcal{C}}|(w) d\lambda(w).$$

16 By theorem (1, § 7, 6). $\forall \lambda w, |f^{\mathcal{C}}|(w) \leq |f|^{\mathcal{C}}(w)$.
Hence

$$\int |f^{\mathcal{C}}|(w) d\lambda(w) \leq \int |f|^{\mathcal{C}}(w) d\lambda(w)$$

$$= \int |f|(w) d\lambda(w).$$

Hence, $\|u_{\mathcal{O}, \mathcal{C}}(f)\| \leq \|f\|$ where $\|u_{\mathcal{O}, \mathcal{C}}(f)\|$ (resp. $\|f\|$) denotes the norm of $u_{\mathcal{O}, \mathcal{C}}(f)$ (resp. norm of f) in $L^1(\Omega; \mathcal{C}; \lambda; E)$ (resp. in $L^1(\Omega; \mathcal{O}; \lambda; E)$).

Hence $u_{\mathcal{O}, \mathcal{C}}$ is a contraction linear map and therefore, there exists a unique extension of this map to the whole of $L^1(\Omega; \mathcal{O}; \lambda; E)$ which we again denote by $u_{\mathcal{O}, \mathcal{C}}$. It can be easily seen that $u_{\mathcal{O}, \mathcal{C}}(f)$ is a conditional expectation of f with respect to \mathcal{C} .

8 A few properties of conditional expectations of Banach space valued integrable functions

As before, let $(\Omega, \mathcal{O}, \lambda)$ be a measure space, \mathcal{C} a σ -algebra contained in $\hat{\mathcal{O}}_\lambda$ and let λ restricted to \mathcal{C} be σ -finite. Let E be a Banach space over the real numbers.

Proposition 9. *If $f \in L^1(\Omega; \mathcal{O}; \lambda; E)$, then*

$$\forall_{\lambda} w, |f^{\mathcal{C}}|(w) \leq |f|^{\mathcal{C}}(w).$$

Proof. There exists a sequence $(f_n)_{n \in \mathbb{N}}$ of step functions belonging to $\hat{\mathcal{O}}_\lambda$ such that

- (i) $f_n \rightarrow f$ in $L^1(\Omega; \mathcal{O}; \lambda; E)$
- (ii) $\forall_{\lambda} w, f_n(w) \rightarrow f(w)$ in E and
- (iii) $\forall_{\lambda} w, f_n^{\mathcal{C}}(w) \rightarrow f^{\mathcal{C}}(w)$ in E .

Since $f_n \rightarrow f$ in $L^1(\Omega; \mathcal{O}; \lambda; E)$, $|f_n| \rightarrow |f|$ in $L^1(\Omega; \mathcal{O}; \lambda)$. Hence $|f_n|^{\mathcal{C}} \rightarrow |f|^{\mathcal{C}}$ in $L^1(\Omega; \mathcal{C}; \lambda)$. Therefore, there exists a subsequence f_{n_k} such that $\forall_{\lambda} w, |f_{n_k}|^{\mathcal{C}}(w) \rightarrow |f|^{\mathcal{C}}(w)$ in E . Since

$$\forall_{\lambda} w, |f_{n_k}^{\mathcal{C}}|(w) \leq |f_{n_k}|^{\mathcal{C}}(w),$$

passing to the limit, we see that

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$$\forall_{\lambda} w, |f^{\mathcal{C}}|(w) \leq |f|^{\mathcal{C}}(w).$$

□

Proposition 10. *Let f_n be a sequence of functions on Ω , belonging to $\hat{\mathcal{O}}_\lambda$ with values in E such that f_n converges to a function f in the sense of the dominated convergence theorem, i.e., $\forall_{\lambda w}, f_n(w) \rightarrow f(w)$ in E and there exists a non-negative real valued λ -integrable function g on Ω such that $\forall_{\lambda w}, \forall n \in \mathbb{N}, |f_n|(w) \leq g(w)$. Then $f_n^{\mathcal{E}}$ converges to $f^{\mathcal{E}}$ also in the sense of the dominated convergence theorem.*

Proof. Without loss of generality let us assume that $f \equiv 0$, and $f_n(w) \rightarrow 0$ for all $w \in \Omega$.

Let $c^\circ(E)$ denote the vector space of all sequences $x = (x_n)_{n \in \mathbb{N}}$, $x_n \in E \forall n \in \mathbb{N}$ and $x_n \rightarrow 0$ in E as $n \rightarrow \infty$. Define $\forall x \in c^\circ(E)$, $\|x\| = \sup_n \|x_n\|$ where $\|x_n\|$ is the norm of the element x_n in E . Then, it is easily seen that $x \rightarrow \|x\|$ is a norm in $c^\circ(E)$ and this norm makes $c^\circ(E)$, a Banach space. \square

Let h be a function on Ω with values in $c^\circ(E)$. Then there exists a sequence $(h_n)_{n \in \mathbb{N}}$ of functions on Ω with values in E such that $\forall w \in \Omega$,

$$h(w) = (h_1(w), h_2(w), \dots, h_n(w), \dots).$$

It can be easily seen that if \mathcal{S} is any σ -algebra on Ω , then $h \in \mathcal{S}$ if and only if $\forall n \in \mathbb{N}, h_n \in \mathcal{S}$ and that if $\mathcal{S} \subset \hat{\mathcal{O}}_\lambda$, then h is λ -integrable if and only if $\forall n \in \mathbb{N}, h_n$ is λ -integrable. Moreover, it can be easily seen that if $h \in \hat{\mathcal{O}}_\lambda$ and is λ -integrable, then $\forall A \in \hat{\mathcal{O}}_\lambda$.

$$\int_A h d\lambda = \left(\int_A h_1 d\lambda, \int_A h_2 d\lambda, \dots, \int_A d\lambda, \dots \right).$$

18 Hence if $h \in \hat{\mathcal{O}}_\lambda$ and is λ -integrable, then

$$\forall_{\lambda w}, h^{\mathcal{E}}(w) = (h_1^{\mathcal{E}}(w), h_2^{\mathcal{E}}(w), \dots, h_n^{\mathcal{E}}(w), \dots)$$

Consider the sequence $(\chi_n)_{n \in \mathbb{N}}$ of functions on Ω with values in $c^\circ(E)$ given by

$$(\chi_n)_m(w) = \begin{cases} 0 & \text{if } m < n \\ f_m(w) & \text{if } m \geq n \end{cases}$$

where $(\chi_n)_m(w)$ stands for the m^{th} coordinate of $\chi_n(w)$.

Then $\forall w \in \Omega$, $\chi_n(w) \rightarrow 0$ in $c^\circ(E)$. $\forall n \in \mathbb{N}$, $\chi_n \in \hat{\mathcal{O}}_\lambda$ and is λ -integrable since $\forall n \in \mathbb{N}$, $\forall_\lambda w$, $|\chi_n|(w) = \sup_{m \geq n} |f_m|(w) \leq g(w)$ and g is λ -integrable.

$$\int |\chi_n| d\lambda = \int \sup_{m \geq n} |f_m|(w) d\lambda(w) \downarrow 0 \text{ as } n \rightarrow \infty.$$

Hence

$$\chi_n \rightarrow 0 \text{ in } L^1(\Omega; \mathcal{O}; \lambda; c^\circ(E)).$$

Therefore,

$$\chi_n^\mathcal{C} \rightarrow 0 \text{ in } L^1(\Omega, \mathcal{C}; \lambda; c^\circ(E)).$$

Let us prove that $\forall_\lambda w$, $\chi_n^\mathcal{C}(w) \rightarrow 0$ in $c^\circ(E)$. Now, $\forall_\lambda w$, $|\chi_n^\mathcal{C}|(w)$ is a decreasing function of n and hence $\lim_{n \rightarrow \infty} |\chi_n^\mathcal{C}|(w)$ exists $\forall_\lambda w$. Let $d_w = \lim_{n \rightarrow \infty} |\chi_n^\mathcal{C}|(w)$ when the limit of $|\chi_n^\mathcal{C}|(w)$ exists.

$$\int |\chi_n^\mathcal{C}|(w) d\lambda(w) \downarrow \int d_w d\lambda(w), \text{ as } n \rightarrow \infty.$$

Since $\chi_n^\mathcal{C} \rightarrow 0$ in $L^1(\Omega; \mathcal{C}; \lambda; c^\circ(E))$, it follows that $\int d_w d\lambda(w) = 0$ and hence $\forall_\lambda w$, $d_w = 0$.

Hence,

$$\forall_\lambda w, \chi_n^\mathcal{C}(w) \rightarrow 0 \text{ in } c^\circ(E).$$

Hence,

$$\forall_\lambda w, \sup_{m \geq n} |f_m^\mathcal{C}|(w) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

This means that $\forall_\lambda w$, $f_n^\mathcal{C}(w) \rightarrow 0$ in E as $n \rightarrow \infty$. Since $|f_n| \leq g$, $\forall_\lambda w$, $|f_n|^\mathcal{C}(w) \leq g^\mathcal{C}(w)$ and $g^\mathcal{C}$ is λ -integrable, since g is. Hence $f_n^\mathcal{C}$ converges to zero, in the sense of the dominated convergence theorem. 19

Remark 11. When $E = \mathbb{R}$, the above proposition (1, § 8, 10) is proved in Doob [1] in the pages 23-24. Though the theory of conditional expectations for Banach space valued functions is not used there as is done here, the idea is essentially the same.

The properties (v), (vi) and (vii) stated in § 5 of this chapter for functions belonging to $L^1(\Omega; \mathcal{O}; \lambda)$ are also true for any $f \in L^1(\Omega; \mathcal{O}; \lambda; E)$, as can be easily seen. Property (iv) also is true with the same assumptions on g and on gf as there.

Thus, in this chapter, we have proved the existence and uniqueness of conditional expectations for Banach space valued λ -integrable functions on a measure space $(\Omega, \mathcal{O}, \lambda)$. We shall see in § 1 of Chapter 3, that the existence of disintegration of λ is linked to the existence of conditional expectations for measure valued functions on Ω , as defined in chapter 2.

Chapter 2

Measure valued Functions

1 Basic definitions: Fubini's theorem for extended real valued integrable functions

In this chapter, we shall study the measure valued functions. 20

Let $(\Omega, \mathcal{O}, \lambda)$ be a measure space. Let (Y, \mathcal{Y}) be a measurable space. Let $m^+(Y, \mathcal{Y})$ be the set of all positive measures on \mathcal{Y} .

Definition 12. A measure valued function ν on Ω with values in $m^+(Y, \mathcal{Y})$ is an assignment to each $w \in \Omega$, a positive measure ν_w on \mathcal{Y} .

If $(\lambda_w^{\mathcal{C}})_{w \in \Omega}$ is a disintegration of λ with respect to a σ -algebra $\mathcal{C} \subset \hat{\mathcal{O}}_\lambda$, it can be considered as a measure valued function $\lambda^{\mathcal{C}}$ on Ω with values in $m^+(\Omega, \mathcal{O})$ taking $w \in \Omega$ to $\lambda_w^{\mathcal{C}}$.

If ν is a measure valued function on Ω with values in $m^+(Y, \mathcal{Y})$ and if f is any non-negative function on Y belonging to \mathcal{Y} , then $\nu(f)$ will denote the function on Ω taking w to $\nu_w(f)$. If B is a set belonging to \mathcal{Y} , $\nu(B)$ will stand for $\nu(\chi_B)$.

Definition 13. A measure valued function ν on Ω with values in $m^+(Y, \mathcal{Y})$ is said to be measurable with respect to a σ -algebra \mathcal{S} on Ω or is said to belong to \mathcal{S} if $\forall B \in \mathcal{Y}$, the extended real valued function $\nu(B)$ belongs to \mathcal{S} .

If ν belongs to \mathcal{S} , we write $\underline{\nu \in \mathcal{S}}$.

If $\lambda^{\mathcal{C}}$ is a disintegration of λ with respect to a σ -algebra $\mathcal{C} \subset \hat{\mathcal{O}}_\lambda$, note that $\lambda^{\mathcal{C}} \in \mathcal{C}$.

21 If $\nu \in \mathcal{S}$, we see immediately that \forall non-negative function f on Y , $f \in \mathcal{Y}$, $\nu(f) \in \mathcal{S}$.

Definition 14. If ν is a measure valued function on Ω with values in $\mathfrak{m}^+(Y, \mathcal{Y})$, belonging to $\hat{\mathcal{O}}_\lambda$, the integral of ν with respect to λ is defined as the measure J on \mathcal{Y} given by $\forall B \in \mathcal{Y}$, $J(B) = \int \nu_w(B) d\lambda(w)$. It is written as $J = \int \nu_w d\lambda(w)$. If $A \in \mathcal{O}$, the integral of ν over A is defined as the integral of ν with respect to the measure $\chi_A \cdot \lambda$. The integral of ν over A is written as $\int_A \nu_w d\lambda(w)$.

Note that if $\lambda^{\mathcal{C}}$ is a disintegration of λ with respect to a σ -algebra $\mathcal{C} \subset \hat{\mathcal{O}}_\lambda$, the integral of $\lambda^{\mathcal{C}}$ with respect to λ is λ . i.e. $\lambda = \int \lambda_w^{\mathcal{C}} d\lambda(w)$.

Note also that from our definition, it easily follows that if J is the integral of ν with respect to λ , then for every function f on Y , $f \geq 0$, $f \in \mathcal{Y}$, $J(f) = \int \nu_w(f) d\lambda(w)$. Hence, $\forall f \geq 0$, $f \in \mathcal{Y}$, $J(f) = 0$ implies that $\forall_\lambda w$, $\nu_w(f) = 0$. In particular, if $B \in \mathcal{Y}$ is such that its J -measure is zero, then $\forall_\lambda w$ its ν_w -measure is also zero.

Note that it also follows easily from our definition that if f is an extended real valued function on Y , $f \in \mathcal{Y}$ and J -integrable, then $\forall_\lambda w$, it is ν_w -integrable. Moreover, the function $w \rightarrow \nu_w(f)$ (defined arbitrarily on the set of points w where f is not ν_w -integrable) belongs to $\hat{\mathcal{O}}_\lambda$ and is λ -integrable. Further, $\int \nu_w(f) d\lambda(w) = J(f)$.

If $A \in \mathcal{O}$, note that $\int_A \nu_w d\lambda(w)(f)$ is equal to $\int_A \nu_w(f) d\lambda(w)$, for every function f on Y , $f \geq 0$ and $f \in \mathcal{Y}$.

The following theorem and the corollary (2, § 1, 16) contains as a special case, as we shall see towards the end of this chapter, the usual Fubini's theorem. Hence we shall call this also as Fubini's theorem.

22 **Theorem 15 (Fubini).** Let \mathcal{S} be a σ -algebra contained in $\hat{\mathcal{O}}_\lambda$. Let ν be a measure valued function on Ω with values in $\mathfrak{m}^+(Y, \mathcal{Y})$ belonging to \mathcal{S} and with integral J . Let f be an extended real valued function on Y , $f \geq 0$ belonging to $\hat{\mathcal{Y}}_J$. Then,

- (i) $\forall_\lambda w$, f is ν_w -measurable, i.e., $f \in \hat{\mathcal{Y}}_{\nu_w}$.

(ii) The function $w \rightarrow \nu_w(f)$ (defined arbitrarily on the set of $w \in \Omega$ for which f is not ν_w -measurable) $\in \hat{\mathcal{S}}_\lambda$ and

$$(iii) \quad \int \nu_w(f) d\lambda(w) = J(f).$$

Proof. (i) Since $f \geq 0$ belongs to $\hat{\mathcal{Y}}_J$, there exist functions f_i on Y , $f_i \geq 0$, $f_i \in \mathcal{Y}$, $i = 1, 2$, such that $f_1 \leq f \leq f_2$ everywhere and the set $B = \{y \in Y \mid f_1(y) \neq f_2(y)\}$ is of J -measure zero. Since $J(B) = 0$, $\forall_\lambda w$, $\nu_w(B) = 0$. i.e., there exists a set $A \in \hat{\mathcal{O}}_\lambda$ such that $\lambda(A) = 0$ and if $w \notin A$, $\nu_w(B) = 0$. Since $f_1 \in \mathcal{Y}$, it follows therefore that if $w \notin A$, f is ν_w -measurable i.e., $f \in \hat{\mathcal{Y}}_{\nu_w}$.

(ii) Since for $i = 1, 2$, $w \rightarrow \nu_w(f_i) \in \mathcal{S}$, it follows that $w \rightarrow \nu_w(f) \in \hat{\mathcal{S}}_\lambda$.

$$(iii) \quad \int \nu_w(f) d\lambda(w) = \int \nu_w(f_i) d\lambda(w) \\ = J(f_i) \\ = J(f)$$

□

Corollary 16 (Fubini). *With \mathcal{S} , ν and J as in the above theorem, if f is a J -integrable extended real valued function on Y , $f \in \hat{\mathcal{Y}}_J$, then*

(i) $\forall_\lambda w$, f is ν_w -measurable, i.e., $f \in \hat{\mathcal{Y}}_{\nu_w}$, and is ν_w -integrable

(ii) the function $w \rightarrow \nu_w(f)$ (defined arbitrarily on the set of $w \in \Omega$ for which f is not ν_w -integrable), belongs to $\hat{\mathcal{S}}_\lambda$ and is λ -integrable, and

$$(iii) \quad \int \nu_w(f) d\lambda(w) = J(f).$$

Proof. If $f = f^+ - f^-$ with the usual notation, then the corollary immediately follows from the above theorem by applying it to f^+ and f^- . □

Corollary 17. *Let \mathcal{C} be a σ -algebra contained in $\hat{\mathcal{O}}_\lambda$. Let $(\lambda_w^\mathcal{C})_{w \in \Omega}$ be a disintegration of λ with respect to \mathcal{C} . Let f be an extended real valued function on Ω , $f \geq 0$ (resp. f λ -integrable) belonging to $\hat{\mathcal{O}}_\lambda$. Then,*

(i) $\forall_\lambda w$, f is $\lambda_w^\mathcal{C}$ -measurable i.e. $f \in \hat{\mathcal{O}}_{\lambda_w^\mathcal{C}}$ (resp. f is $\lambda_w^\mathcal{C}$ -measurable and $\lambda_w^\mathcal{C}$ -integrable).

- (ii) *the function $w \rightarrow \lambda_w^{\mathcal{C}}(f)$ (defined arbitrarily on the set of $w \in \Omega$, for which f is not $\lambda_w^{\mathcal{C}}$ -measurable (resp. f is not $\lambda_w^{\mathcal{C}}$ -integrable)) belongs to $\hat{\mathcal{C}}_\lambda$ and is λ -integrable and*
- (iii)
$$\int \lambda_w^{\mathcal{C}}(f) d\lambda(w) = \lambda(f).$$

Proof. When f is ≥ 0 (resp. f is λ -integrable), this follows immediately from theorem (2, § 1, 15) (resp. from the above Corollary (2, § 1, 16)) by taking $\lambda^{\mathcal{C}}$ instead of ν and observing that $\lambda^{\mathcal{C}} \in \mathcal{C}$ and $\int \lambda_w^{\mathcal{C}} d\lambda(w) = \lambda$. \square

We saw in § 3 of Chapter 1, how the existence of a disintegration $(\lambda_w^{\mathcal{C}})_{w \in \Omega}$ of λ with respect to \mathcal{C} implies immediately the existence of conditional expectations for functions $f \geq 0$ on Ω , $f \in \mathcal{O}$ and that $w \rightarrow \lambda_w^{\mathcal{C}}(f)$ is a conditional expectation of f with respect to \mathcal{C} . If f is ≥ 0 on Ω and belongs to $\hat{\mathcal{O}}_\lambda$, we see from the above Corollary (2, § 1, 17) that $\forall \lambda w$, f is $\lambda_w^{\mathcal{C}}$ -measurable and the function $w \rightarrow \lambda_w^{\mathcal{C}}(f)$ belongs to $\hat{\mathcal{C}}_\lambda$. It can be easily checked that the function $w \rightarrow \lambda_w^{\mathcal{C}}(f)$ is actually a conditional expectation of f with respect to $\hat{\mathcal{C}}_\lambda$ and hence is almost everywhere equal to a conditional expectation of f with respect to \mathcal{C} . Similar result holds again when $f \in \hat{\mathcal{O}}_\lambda$ and is λ -integrable.

Thus, we see how the existence of a disintegration of λ with respect to a σ -algebra \mathcal{C} contained in $\hat{\mathcal{O}}_\lambda$, implies as well the existence of conditional expectations with respect to \mathcal{C} for λ -integrable extended real valued functions belonging to $\hat{\mathcal{O}}_\lambda$.

24 In § 2, we shall extend the results of this section to Banach space valued λ -integrable functions.

2 Fubini's theorem for Banach space valued integrable functions

Throughout this section, we fix a measure space $(\Omega, \mathcal{O}, \lambda)$, a σ -algebra $\mathcal{S} \subset \hat{\mathcal{O}}_\lambda$, a measurable space (Y, \mathcal{Y}) , a measure valued function ν on Ω with values in $m^+(Ym, \mathcal{Y})$ belonging to \mathcal{S} , having an integral J with respect to λ and a Banach space E over the real numbers.

Proposition 18. *Let g be a step function on Y with values in E belonging to $\hat{\mathcal{Y}}_J$ (resp. \mathcal{Y}). Let further, g be J -integrable.*

Then

- (i) $\forall \lambda w, g \in \hat{\mathcal{Y}}_{\nu_w}$ and is ν_w -integrable (resp. $\forall \lambda w, g$ is ν_w -integrable).
- (ii) *The function $w \rightarrow \nu_w(g)$ with values in E (defined arbitrarily, on the set of points $w \in \Omega$ where g is not ν_w -integrable, in such a way that $\forall w, \nu_w(g)$ is still an element of E) belongs to $\hat{\mathcal{S}}_\lambda$ (resp. belongs to \mathcal{S}) and is λ -integrable and*
- (iii) $\int \nu_w(g) d\lambda(w) = J(g).$

Proof. (i) Let $g = \sum_{i=1}^n \chi_{A_i} \cdot x_i$ where for $i = 1, \dots, n, A_i \in \hat{\mathcal{Y}}_J$ (resp. $A_i \in \mathcal{Y}$), $A_i \cap A_j = \emptyset$ if $i \neq j$ and $x_i \in E$. By theorem (2, § 1, 15), $\forall i, \forall \lambda w, A_i \in \hat{\mathcal{Y}}_{\nu_w}$. Hence $\forall \lambda w, \forall i, A_i \in \hat{\mathcal{Y}}_{\nu_w}$. Therefore, $\forall \lambda w, g \in \hat{\mathcal{Y}}_{\nu_w}$.

Since g is J -integrable, $\forall i, J(A_i) < +\infty$. Hence $\forall \lambda w, \forall i, \nu_w(A_i) < +\infty$. Hence $\forall \lambda w, g$ is ν_w -integrable.

- (ii) Let $A = \{w \in \Omega \mid g \text{ is } \nu_w \text{-integrable}\}$. By (i) λ is carried by A and $\forall w \in A,$

$$\nu_w(g) = \sum_{i=1}^n \nu_w(A_i) x_i$$

$\forall n \in \mathbb{N}, \forall w \in \Omega,$ let $g_n(w) = \sum_{i=1}^n \inf(\nu_w(A_i), n) x_i.$

Then, $\forall n \in \mathbb{N} g_n$ is a finite dimensional valued function belonging to $\hat{\mathcal{S}}_\lambda$ (resp. belonging to \mathcal{S}). Further, $g_n(w) \rightarrow \nu_w(g)$ in $E \forall w \in A$. Since λ is carried by A and since, $\forall n \in \mathbb{N} g_n \in \hat{\mathcal{S}}_\lambda$ (resp. $g_n \in \mathcal{S}$ and $A \in \mathcal{S}$) it follows that the function $w \rightarrow \nu_w(g)$ belongs to $\hat{\mathcal{S}}_\lambda$ (resp. belongs to \mathcal{S}).

$\forall i, w \rightarrow \nu_w(A_i)$ is λ -integrable, since $J(A_i) < +\infty$, and hence $w \rightarrow \nu_w(g)$ is λ -integrable, since for $w \in A,$

$$|\nu_w(g)| \leq \sum_{i=1}^n |\nu_w(A_i)| \cdot \|x_i\|.$$

$$\begin{aligned}
\text{(iii)} \quad \int \nu_w(g) d\lambda(w) &= \int \left(\sum_{i=1}^n \nu_w(A_i) x_i \right) d\lambda(w) \\
&= \sum_{i=1}^n \int \nu_w(A_i) d\lambda(w) \cdot x_i \\
&= \sum_{i=1}^n J(A_i) x_i \\
&= J(g)
\end{aligned}$$

□

Corollary 19. Let \mathcal{C} be a σ -algebra contained in $\hat{\mathcal{O}}_\lambda$ and let $(\lambda_w^\mathcal{C})_{w \in \Omega}$ be a disintegration of λ with respect to \mathcal{C} . Let g be a step function on Ω with values in E , $g \in \hat{\mathcal{O}}_\lambda$ (resp. $g \in \mathcal{O}$) and λ -integrable. Then

- (i) $\forall \lambda w, g \in \hat{\mathcal{O}}_{\lambda_w^\mathcal{C}}$ and is λ -integrable (resp. $\forall \lambda w, g$ is $\lambda_w^\mathcal{C}$ -integrable).
- (ii) The function $w \rightarrow \lambda_w^\mathcal{C}(g)$ with values in E (defined arbitrarily, on the set of points $w \in \Omega$ where g is not $\lambda_w^\mathcal{C}$ -integrable, in such a way that $\forall w \in \Omega, \lambda_w^\mathcal{C}(g)$ is still an element of E) belongs to $\hat{\mathcal{C}}_\lambda$ (resp. belongs to \mathcal{C}) and is λ -integrable and
- (iii) $\int \lambda_w^\mathcal{C}(g) d\lambda(w) = \lambda(g)$.

Proof. This follows immediately from the above proposition (2, § 2, 18), by taking $\lambda^\mathcal{C}$ instead of ν and observing that $\lambda^\mathcal{C} \in \mathcal{C}$ and $\int \lambda_w^\mathcal{C} d\lambda(w) = \lambda$. □

26 In the following theorem, let us consider the case of an arbitrary J -integrable function on Y with values in E . Since this theorem contains as a special case, the usual Fubini's theorem, as we shall see below, we call it also as Fubini's theorem.

Theorem 20 (Fubini). Let f be a function on Y with values in E , $f \in \hat{\mathcal{Y}}_J$ (resp. $f \in \mathcal{Y}$) and J -integrable. Then,

- (i) $\forall \lambda w, f \in \hat{\mathcal{Y}}_{\nu_w}$ and is ν_w -integrable (resp. $\forall \lambda w, f$ is ν_w -integrable).
- (ii) The function $w \rightarrow \nu_w(f)$ on Ω with values in E (defined arbitrarily on the set of points $w \in \Omega$ where f is not ν_w -integrable) belongs to $\hat{\mathcal{S}}_\lambda$ (resp. belongs to \mathcal{S}) and is λ -integrable and

$$(iii) \quad \int \nu_w(f) d\lambda(w) = J(f).$$

Proof. Since $f \in \hat{\mathcal{Y}}_J$ (resp. \mathcal{Y}) and is J -integrable, there exists a sequence $(g_n)_{n \in \mathbb{N}}$ of step functions on Y with values in E and a non-negative real valued function g on Y belonging to \mathcal{Y} and J -integrable such that $\forall n, g_n \in \hat{\mathcal{Y}}_J$ (resp. $\forall n, g_n \in \mathcal{Y}$) and J -integrable, $\forall Jy, |g_n|(y) \leq g(y), \forall Jy, g_n(y) \rightarrow f(y)$ in E and $J(g_n) \rightarrow J(f)$ in E .

- (i) By proposition (2, § 2, 18), $\forall n, \forall \lambda w, g_n \in \hat{\mathcal{Y}}_{\nu_w}$. Since $\forall Jy, g_n(y) \rightarrow f(y)$ in $E, \forall \lambda w, \forall \nu_w y, g_n(y) \rightarrow f(y)$ in E . Hence, $\forall \lambda w, f \in \hat{\mathcal{Y}}_{\nu_w}$. Further, by the same proposition (2, § 2, 18), $\forall n, \forall \lambda w, g_n$ is ν_w -integrable. Hence, $\forall \lambda w, \forall n, g_n$ is ν_w -integrable.

Also $\forall \lambda w, g$ is ν_w -integrable.

Since $\forall Jy, \forall n, |g_n|(y) \leq g(y)$, we have $\forall \lambda w, \forall \nu_w y, \forall n, |g_n|(y) \leq g(y)$ and hence,

$$\forall \lambda w, \forall n \in \mathbb{N}, \nu_w(|g_n|) \leq \nu_w(g) < +\infty.$$

Hence, by Fatou's lemma, $\forall \lambda w, f$ is ν_w -integrable, and by the dominated convergence theorem,

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$$\forall \lambda w, \nu_w(g_n) \rightarrow \nu_w(f) \text{ in } E.$$

- (ii) By proposition (2, § 2, 18), $\forall n \in \mathbb{N}$, the function $w \rightarrow \nu_w(g_n)$ belongs to $\hat{\mathcal{S}}_\lambda$ (resp. belongs to \mathcal{S}) and since $\forall \lambda w, \nu_w(g_n) \rightarrow \nu_w(g)$ in E (resp. since the set of w where $\nu_w(g_n)$ converges to $\nu_w(f)$ belongs to \mathcal{S} and carries λ) it follows that $w \rightarrow \nu_w(f)$ also belongs to $\hat{\mathcal{S}}_\lambda$ (resp. belongs to \mathcal{S}).

Since $\forall \lambda w, \nu_w(g_n) \rightarrow \nu_w(f)$ in E and since $\forall \lambda w, \forall n \in \mathbb{N}, |\nu_w(g_n)| \leq \nu_w(|g_n|) \leq \nu_w(g)$ and since $w \rightarrow \nu_w(g)$ is λ -integrable, it follows by Fatou's lemma again that $w \rightarrow \nu_w(f)$ is λ -integrable and again by the dominated convergence theorem,

$$\int \nu_w(g_n) d\lambda(w) \rightarrow \int \nu_w(f) d\lambda(w).$$

(iii) $J(g_n) \rightarrow J(f)$ in E .

But $\forall n \in \mathbb{N}$, $J(g_n) = \int \nu_w(g_n)d\lambda(w)$ by proposition (2, § 2, 18). Since $\int \nu_w(g_n)d\lambda(w)$ converges to $\int \nu_w(f)d\lambda(w)$, it follows that

$$J(f) = \int \nu_w(f)d\lambda(w).$$

□

Corollary 21. Let \mathcal{C} be a σ -algebra contained in $\hat{\mathcal{O}}_\lambda$, and let $(\lambda_w^\mathcal{C})_{w \in \Omega}$ be a disintegration of λ with respect to \mathcal{C} . Let f be a function on Ω with values in E , $f \in \hat{\mathcal{O}}_\lambda$ (resp. $f \in \mathcal{O}$) and λ -integrable. Then

(i) $\forall \lambda w, f \in \hat{\mathcal{O}}_{\lambda_w^\mathcal{C}}$ and is $\lambda_w^\mathcal{C}$ -integrable

(ii) The function $w \rightarrow \lambda_w^\mathcal{C}(f)$ with values in E (defined arbitrarily on the set of $w \in \Omega$ where f is not $\lambda_w^\mathcal{C}$ -integrable in such a way that $\lambda_w^\mathcal{C}(f)$ still takes values in $E \forall w \in \Omega$) belongs to $\hat{\mathcal{C}}_\lambda$ (resp. belongs to \mathcal{C}) and is λ -integrable. and

(iii) $\int \lambda_w^\mathcal{C}(f)d\lambda(w) = \lambda(f)$.

28 *Proof.* This follows immediately from the above theorem (2, § 2, 20) by applying it to $\lambda^\mathcal{C}$ instead of ν and observing that $\lambda^\mathcal{C} \in \mathcal{C}$ and $\int \lambda_w^\mathcal{C} d\lambda(w) = \lambda$. □

We shall now deduce the usual Fubini's theorem from the above theorem (2, § 2, 20).

Let (X, \mathfrak{X}, μ) and (Z, \mathfrak{Z}, ν) be two measure spaces with μ (resp. ν) σ -finite on \mathfrak{X} (resp. on \mathfrak{Z}). Let $\forall x \in X$, $\delta_x \otimes \nu$ be the product measure of δ_x and ν on the σ -algebra $\mathfrak{X} \otimes \mathfrak{Z}$ on the set $X \times Z$. Then $x \rightarrow \delta_x \otimes \nu$ is a measure valued function on X with values in $m^+(X \times Z, \mathfrak{X} \otimes \mathfrak{Z})$, belonging to \mathfrak{X} and has the measure $\mu \otimes \nu$ for its integral with respect to μ .

Let f be a function on $X \times Z$ with values in a Banach space E , $f \in \widehat{\mathfrak{X} \otimes \mathfrak{Z}}_{\mu \otimes \nu}$. Then, by the above theorem (2, § 2, 20).

- (i) $\forall \lambda x, f \in \mathfrak{X}_{\hat{\mathfrak{z}}_{\delta_x \otimes \nu}}$ and is $\delta_x \otimes \nu$ integrable; i.e., $\forall \mu x$, the function $z \rightarrow f(x, z)$ belongs to $\hat{\mathfrak{z}}_\nu$ and is ν -integrable and

$$\forall \nu x, \int |f|(x, z) d\nu(z) < +\infty.$$

- (ii) $x \rightarrow \delta_x \otimes \nu(f)$ belongs to $\hat{\mathfrak{X}}_\mu$ and is μ -integrable i.e., $x \rightarrow \int f(x, z) d\nu(z)$ is μ -integrable and

$$(iii) \quad \int (\delta_x \otimes \nu)(f) d\mu(x) = \int f d\mu \otimes \nu$$

$$i.e. \int \left(\int f(x, z) d\nu(z) \right) d\mu(x) = \int f(x, z) d\mu \otimes \nu(x, z).$$

When $E = \mathbb{R}$, this is the usual Fubini's theorem.

It is now clear how to deduce the Fubini's theorem for functions which are non-negative (resp. integrable) and extended real valued from theorem (2, § 1, 15) (resp. Corollary (2, § 1, 16)).

Form Corollary (2, § 2, 21), we see that if f is a J -integrable function on Y with values in a Banach space E , $f \in \mathfrak{Y}_J$ (resp. $f \in \mathfrak{Y}$) and if \mathcal{C} is a σ -algebra contained in $\hat{\mathcal{O}}_\lambda$ and if $(\lambda_w^\mathcal{C})_{w \in \Omega}$ is a disintegration of λ with respect to \mathcal{C} , then the almost everywhere defined function $w \rightarrow \lambda_w^\mathcal{C}(f)$ belongs to $\hat{\mathcal{C}}_\lambda$ (resp. \mathcal{C}) and is λ -integrable. One can easily check that $w \rightarrow \lambda_w^\mathcal{C}(f)$ is actually a conditional expectation of f with respect to $\hat{\mathcal{C}}_\lambda$ (resp. with respect to \mathcal{C}) first by considering step functions and then extending to f . Thus $w \rightarrow \lambda_w^\mathcal{C}(f)$ is almost everywhere equal to a conditional expectation of f with respect to \mathcal{C} (resp. is a conditional expectation of f with respect to \mathcal{C}).

Thus, we see how the existence of a disintegration implies the existence of conditional expectations for Banach space valued integrable functions as well. Hence the importance of the existence of disintegrations. In the next chapter, we shall give some sufficient conditions for the existence of disintegration of a measure with respect to a σ -algebra.

Chapter 3

Conditional expectations of measure valued functions, Existence and uniqueness theorems

1 Basic definition

In this chapter, we shall define the notion of conditional expectation 30
for measure valued functions, and prove some existence and uniqueness
theorems. Our results, as we shall see, will give immediately an im-
portant consequence of a result of M. Jirina [1] on regular conditional
probabilities.

Let $(\Omega, \mathcal{O}, \lambda)$ be a measure space. Let \mathcal{C} be a σ -algebra contained
in $\hat{\mathcal{O}}_\lambda$. Let λ restricted to \mathcal{C} be σ -finite. Let (Y, \mathcal{Y}) be a measurable
space. Let ν be a measure valued function on Ω with values in $m^+(Y, \mathcal{Y})$,
 $\nu \in \hat{\mathcal{O}}_\lambda$.

Definition 22. A measure valued function $\nu^\mathcal{C}$ on Ω with values in $m^+(Y, \mathcal{Y})$
is said to be a conditional expectation of ν with respect to \mathcal{C} if

- (i) $\nu^\mathcal{C} \in \mathcal{C}$ and

$$(ii) \quad \forall A \in \mathcal{C}, \int_A v_w^{\mathcal{C}} d\lambda(w) = \int_A v_w d\lambda(w).$$

Note that $v^{\mathcal{C}}$ is a conditional expectation of v if and only if for every function f on Y , $f \geq 0$, $f \in \mathcal{Y}$, the function $v^{\mathcal{C}}(f) \in \mathcal{C}$ and

$$\forall A \in \mathcal{C}, \int_A v_A^{\mathcal{C}}(f) d\lambda(w) = \int_A v_w(f) f \lambda(w).$$

Thus, $v^{\mathcal{C}}$ is a conditional expectation of v if and only if for every function f on Y , $f \geq 0$, $f \in \mathcal{Y}$, the function $v^{\mathcal{C}}(f)$ is a conditional expectation of $v(f)$.

We note also that if $(\lambda_w^{\mathcal{C}})_{w \in \Omega}$ is a disintegration of λ with respect to \mathcal{C} , then the measure valued function $\lambda^{\mathcal{C}}$ on Ω taking $w \in \Omega$ to $\lambda_w^{\mathcal{C}}$ is a conditional expectation with respect to \mathcal{C} of the measure valued function δ on Ω with values in $m^+(\Omega, \mathcal{O})$, taking $w \in \Omega$ to the measure δ_w (the Dirac measure at w). And conversely, every conditional expectation of the measure valued function δ is a disintegration of λ . Thus, we see how the existence of disintegration for a measure is related to the existence of conditional expectation of measure valued functions.

2 Preliminaries

Before we proceed to prove the existence and uniqueness theorems of conditional expectations for measure valued functions, we collect below in the subsections § 2.1, § 2.2, § 2.3 and § 2.4, some important theorems, propositions and definitions which will be used in the proofs of the main theorems of this chapter.

2.1 The monotone class theorem and its consequences

Let X be a non-void set. Let \mathcal{S} be a class of subsets of X .

We say \mathcal{S} is a π -system on X if it is closed with respect to finite intersections.

We say \mathcal{S} is a d -system on X if

- (i) $X \in \mathcal{S}$,

(ii) $A, B \in \mathcal{S}, A \subset B \Rightarrow B \setminus A \in \mathcal{S}$ and

(iii) $\forall n \in \mathbb{N}, A_n \in \mathcal{S}, A_n \uparrow \Rightarrow \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{S}$.

If \mathcal{C} is any class of subsets of X , we shall denote by $d(\mathcal{C})$ (resp. $\sigma(\mathcal{C})$) the smallest d -system (resp. σ -algebra) containing \mathcal{C} . $d(\mathcal{C})$ (resp. $\sigma(\mathcal{C})$) will be called the d -system (resp. σ -algebra) generated by \mathcal{C} .

In these Notes, the following theorem, whenever it is referred to will always be referred to as the *Monotone class theorem*.

Theorem (Monotone class theorem) *If \mathcal{S} is a π -system, $d(\mathcal{S}) = \sigma(\mathcal{S})$.*

For a proof of this theorem, see R.M. Blumenthal and R.K. Gettoor [1]. Chap. 0, § 2, page 5, theorem (2.2).

Proposition 23. *Let (X, \mathfrak{R}) be a measurable space. Let \mathcal{S} be a π -system containing X and generating \mathfrak{X} . Let μ and ν be two positive finite measures on \mathfrak{X} . If μ and ν agree on \mathcal{S} , then μ and ν are equal.* 32

Proof. The class $\mathcal{C} = \{A \in \mathfrak{X} \mid \mu(A) = \nu(A)\}$ is a d -system containing \mathcal{S} . Hence $\mathcal{C} \supset d(\mathcal{S})$. But by the Monotone class theorem, $d(\mathcal{S})$ is the σ -algebra generated by \mathcal{S} which is \mathfrak{X} . Hence $\mathcal{C} \supset \mathfrak{X}$ and therefore $\mathcal{C} = \mathfrak{X}$ and thus μ and ν are equal. \square

Proposition 24. *Let μ and ν be two positive σ -finite measures on a measurable space (X, \mathfrak{X}) . Let $\mathcal{S} \subset \mathfrak{X}$ be a π -system generating \mathfrak{X} and containing a sequence $(B_n)_{n \in \mathbb{N}}$ with $\bigcup_{n \in \mathbb{N}} B_n = X, \mu(B_n) = \nu(B_n) < +\infty \forall n \in \mathbb{N}$. If μ and ν agree on \mathcal{S} , then μ and ν are equal.*

Proof. First, let us fix a $n \in \mathbb{N}$.

Consider the class $\mathcal{C} = \{A \in \mathfrak{X} \mid \mu(A \cap B_n) = \nu(A \cap B_n)\}$. Then \mathcal{C} is a d -system containing \mathcal{S} . Since \mathcal{S} is a π -system and generates \mathfrak{X} , by the Monotone class theorem $\mathcal{C} \subset \mathfrak{X}$. Hence,

$$\forall A \in \mathfrak{X}, \mu(A \cap B_n) = \nu(A \cap B_n).$$

Since n is arbitrary, we have

$$\forall n \in \mathbb{N}, \forall A \in \mathfrak{X}, \mu(A \cap B_n) = \nu(A \cap B_n).$$

if $A_n = \bigcup_{i=1}^n B_i$, note that

$$\forall n, \forall A \in \mathfrak{X}, \mu(A \cap A_n) = \nu(A \cap A_n).$$

Since

$$A_n \uparrow X, \text{ we see that } \forall A \in \mathfrak{X}, \mu(A) = \nu(A).$$

□

2.2 The lifting theorem of D. Maharam

33 Let (X, \mathfrak{X}, μ) be a measure space with μ σ -finite on \mathfrak{X} . Let $L^\infty(X; \mathfrak{X}; \mu)$ stand for the Banach space whose underlying vector space is the vector space of all μ -equivalence classes of extended real valued functions on X , belonging to $\hat{\mathfrak{X}}_\mu$, having a finite essential supremum and the norm is the essential supremum. Let $\mathcal{B}(X; \mathfrak{X})$ stand for the Banach space whose underlying vector space is the vector space of all real valued bounded functions on X belonging to $\hat{\mathfrak{X}}_\mu$ and the norm is the supremum.

In the following, we have to keep in mind the fact that when we say f is an element of $L^\infty(X; \mathfrak{X}; \mu)$, we mean by f not a single function on X , but a class of functions on X , any two functions of a class differing only on a set of μ -measure zero at most. Thus if g is a function on X and $f \in L^\infty(X; \mathfrak{X}; \mu)$, the meaning of ' $g \in f$ ' is clear i.e. g belongs to the class f . If h is a bounded function on X , $h \in \hat{\mathfrak{X}}_\mu$, $\overline{h} \in$ will denote the unique element of $L^\infty(X; \mathfrak{X}; \mu)$ to which h belongs. Thus, if ' a ' is a real number, considered as the constant function ' a ' on X , the meaning of \underline{a} is clear. If $f \in L^\infty(X; \mathfrak{X}; \mu)$, we say f is *non-negative* and write $\underline{f} \geq 0$ if there exists a function g on X , $g \in f$ and a set $N_g \in \hat{\mathfrak{X}}_\mu$ with $\mu(N_g) = 0$ such that $g \geq 0$ on $\complement N_g$. Note that if there exists one function on $g \in f$ having this property viz. there exists a set N_g with $\mu(N_g) = 0$ and $g \geq 0$ on $\complement N_g$, then every function belonging to f also has this property. If f_1 and f_2 are two elements $\in L^\infty(X; \mathfrak{X}; \mu)$, we say f_1 is *greater than or equal* to f_2 and write $f_1 \geq f_2$, if $f_1 - f_2 \geq 0$.

The following important theorem is due to D. Maharam and whenever we refer this theorem, we will be referring it as the *lifting theorem*. For a proof of this theorem see D. Maharam [1].

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Theorem D. Maharam The Lifting Theorem. *There exists a mapping ρ from $L^\infty(X; \mathfrak{X}; \mu)$ to $\mathcal{B}(X; \mathfrak{X})$ such that*

- (i) ρ is linear and continuous
- (ii) $\rho(f) \in f \quad \forall f \in L^\infty(X; \mathfrak{X}; \mu)$
- (iii) $\rho(f) \geq 0$ if $f \geq 0$ and
- (iv) $\rho(\underline{1}) \equiv 1$

Such a mapping ρ is called a *lifting*.

2.3 Some theorems on $L^\infty(X; \mathfrak{X}; \mu)$

In this section, let $(X; \mathfrak{X}, \mu)$ be a measure space with $\mu(X) < +\infty$.

If $f \in L^\infty(X; \mathfrak{X}; \mu)$, we define the integral of f with respect to μ as $\int g d\mu$ where g is any function belonging to f . Note that $\int g d\mu$ exists since μ is a finite measure and g is essentially bounded. Note also that the integral of f with respect to μ is independent of the choice of the function g chosen to belong to f . The integral of f with respect to μ is written as $\int f d\mu$.

Definition 25. Let $(f_i)_{i \in I}$ be a family of elements of $L^\infty(X; \mathfrak{X}; \mu)$. An element $f \in L^\infty(X; \mathfrak{X}; \mu)$ is said to be a *supremum* of the family $(f_i)_{i \in I}$ in the L^∞ -sense if

- (i) $f \geq f_i \forall i \in I$ and
- (ii) $g \in L^\infty(X; \mathfrak{X}; \mu), g \geq f_i \quad \forall i \in I \Rightarrow g \geq f$.

Note that through a supremum need not always exist, it is unique if it exists.

If the supremum of a family $(f_i)_{i \in I}, f_i \in L^\infty(X; \mathfrak{X}; \mu) \quad \forall i$, exists in the L^∞ -sense, we say $\text{SUP}_{i \in I} f_i$ exists and denote the supremum by $\text{SUP}_{i \in I} f_i$.

If I is a finite set say $\{1, \dots, n\}$, then we write $\text{SUP}_\mu(f_1, \dots, f_n)$ instead of $\text{SUP}_{i \in \{1, \dots, n\}} \mu f_i$. If $f \in L^\infty(X; \mathfrak{X}; \mu)$, we denote by $\|f\|_\infty$, the essential supremum of f .

Proposition 26. *If $(f_n)_{n \in \mathbb{N}}$ is an increasing sequence of elements $\in L^\infty(X; \mathfrak{X}; \mu)$ such that it is bounded in norm i.e. $\sup_n \|f_n\|_\infty < +\infty$, then*

$$35 \quad \text{SUP}_{n \in \mathbb{N}} \mu f_n \text{ exists and } \int (\text{SUP}_\mu f_n) d\mu = \sup_{n \in \mathbb{N}} \int f_n d\mu.$$

Proof. Choose $\forall n \in \mathbb{N}$, a function h_n on X , $h_n \in \hat{\mathfrak{X}}_\mu$ such that $h_n \in f_n$. Since $(f_n)_{n \in \mathbb{N}}$ is increasing, $\forall n \in \mathbb{N}$, \exists a set $E_n \in \hat{\mathfrak{X}}_\mu$ such that $\mu(E_n) = 0$ and if $x \notin E_n$, $h_n(x) \leq h_{n+1}(x)$.

Let $E = \bigcup_{n=1}^{\infty} E_n$. Then $E \in \hat{\mathfrak{X}}_\mu$ and $\mu(E) = 0$. If $x \notin E$, $h_1(x) \leq h_2(x) \leq \dots \leq h_n(x) \leq h_{n+1}(x) \leq \dots$. Thus, if $x \notin E$, $(h_n(x))_{n \in \mathbb{N}}$ is a monotonic non-decreasing sequence of extended real numbers and hence, $\forall x \notin E$, $\lim_{n \rightarrow \infty} h_n(x)$ exists.

Define

$$h(x) = \begin{cases} \lim_{n \rightarrow \infty} h_n(x) & \text{if } x \notin E \\ 0 & \text{if } x \in E. \end{cases}$$

Then $h \in \hat{\mathfrak{X}}_\mu$ since $\sup_n \|f_n\|_\infty < +\infty$.

It is clear that h is the supremum of $(f_n)_{n \in \mathbb{N}}$ in the L^∞ -sense.

Since $\int h d\mu = \lim_{n \rightarrow \infty} \int h_n d\mu$, it follows that

$$\int (\text{SUP}_{n \in \mathbb{N}} \mu f_n) d\mu = \sup_{n \in \mathbb{N}} \int f_n d\mu.$$

□

Proposition 27. *Let $(f_i)_{i \in I}$ be a directed increasing family of elements of $L^\infty(X; \mathfrak{X}; \mu)$ such that the family is bounded in norm, i.e. $\sup_{i \in I} \|f_i\|_\infty < +\infty$. Then, $\text{SUP}_{i \in I} \mu f_i$ exists.*

Proof. Since μ is a finite measure and since

$$\sup_{i \in I} \|f_i\|_\infty < +\infty, \quad \sup_{i \in I} \int f_i d\mu < +\infty.$$

Let $\alpha = \sup_{i \in I} \int f_i d\mu$. Then $\alpha \in \mathbb{R}$.

Since $(f_i)_{i \in I}$ is a directed increasing family, we can find an increasing sequence $(f_n)_{n \in \mathbb{N}}$ such that $\forall n \in \mathbb{N}$, f_n belongs to the family $(f_i)_{i \in I}$, and $\int f_n d\mu \uparrow \alpha$. From the previous proposition, $\text{SUP}_{\mu} f_n$ exists. Let $f = \text{SUP}_{\mu} f_n$.

Let us show that f is the supremum of $(f_i)_{i \in I}$ in the L^∞ -sense. 36

Fix an $i \in I$.

It is easily seen that

$$\int \text{SUP}_{\mu}(f_i, f_n) d\mu \uparrow \int \text{SUP}_{\mu}(f_i, f) d\mu \text{ as } n \rightarrow \infty.$$

$$\forall n \in \mathbb{N}, \int \text{SUP}_{\mu}(f_i, f_n) d\mu \leq \sup_{j \in I} \int f_j d\mu,$$

since $(f_i)_{i \in I}$ is a directed increasing family.

Hence,

$$\forall n \in \mathbb{N}, \int \text{SUP}_{\mu}(f_i, f_n) d\mu \leq \alpha.$$

Hence,

$$\int \text{SUP}_{\mu}(f_i, f) d\mu \leq \alpha.$$

But $\int \text{SUP}_{\mu}(f_i, f) d\mu \geq \alpha$, since $\text{SUP}_{\mu}(f_i, f) \geq f$ and $\int f d\mu = \alpha$.

Hence,

$$\int \text{SUP}_{\mu}(f_i, f) d\mu = \alpha = \int f d\mu.$$

This shows that $f \geq f_i$.

Since $i \in I$ is arbitrary, it follows that $f \geq f_i \quad \forall i \in I$. Now, let $g \in L^\infty(X; \mathfrak{X}; \mu)$ be such that $g \geq f_i \quad \forall i \in I$. Then $g \geq f_n \quad \forall n \in \mathbb{N}$ and hence $g \geq f$.

This shows that f is the supremum of $(f_i)_{i \in I}$ in the L^∞ -sense. □

Proposition 28. Let $(f_i)_{i \in I}$ be a directed increasing family of elements of $L^\infty(X; \mathfrak{X}; \mu)$ and let $f \in L^\infty(X; \mathfrak{X}; \mu)$. Then, the following are equivalent,

(i) $f = \text{SUP}_{\mu} f_i$

(ii) $f \geq f_i \quad \forall i \in I$ and there exists a sequence $(f_n)_{n \in \mathbb{N}}, \forall n \in \mathbb{N} f_n$ belonging to the family $(f_i)_{i \in I}$ such that $f = \text{SUP}_{n \in \mathbb{N}} \mu f_n$.

(iii) $f \geq f_i \quad \forall i \in I$ and $\int f d\mu = \sup_{i \in I} \int f_i d\mu$.

Proof. (i) \implies (ii). Since $f = \text{SUP}_{i \in I} \mu f_i$ and $f \in L^\infty(X; \mathfrak{X}; \mu)$, it follows that the family $(f_i)_{i \in I}$ is bounded in norm. Let $\alpha = \sup_{i \in I} \int f_i d\mu$.

37 Then $\alpha \in \mathbb{R}$. Then, if $(f_n)_{n \in \mathbb{N}}$ is an increasing sequence of elements, $\forall n \in \mathbb{N}, f_n$ belonging to the family $(f_i)_{i \in I}$ such that $\int f_n d\mu \uparrow \alpha$, we can show as in the proof of the previous proposition (3, § 2.3, 27) that $f = \text{SUP}_{n \in \mathbb{N}} \mu f_n$.

(ii) \implies (iii). Let $h_n = \text{SUP}_{\mu}(f_1, f_2, \dots, f_n)$. Then $\forall n \in \mathbb{N}, h_n \in L^\infty(X; \mathfrak{X}; \mu)$ and $\int h_n d\mu \uparrow \int f d\mu$.

Since $\forall i \in I, f \geq f_i, \int f d\mu \geq \int f_i d\mu \quad \forall i \in I$, and hence, $\int f d\mu \geq \sup_{i \in I} \int f_i d\mu$.

On the other hand, $\forall n \in \mathbb{N}, \int h_n d\mu \leq \sup_{i \in I} \int f_i d\mu$, (since the family $(f_i)_{i \in I}$ is directed increasing).

Hence

$$\int f d\mu = \sup_{i \in I} \int f_i d\mu.$$

(iii) \implies (i). Since $f \in L^\infty(X; \mathfrak{X}; \mu)$ and $f \geq f_i \quad \forall i \in I$, the family $(f_i)_{i \in I}$ is bounded in norm.

Let $\alpha = \int f d\mu = \sum_{i \in I} \int f_i d\mu$. Then $\alpha \in \mathbb{R}$. Let $(g_n)_{n \in \mathbb{N}}$ be an increasing sequence of elements belonging to $L^\infty(X; \mathfrak{X}; \mu)$ such that $\forall n \in \mathbb{N}, g_n$ belongs to the family $(f_i)_{i \in I}$ and $\int g_n d\mu \downarrow \alpha$.

Let $g = \text{SUP}_{n \in \mathbb{N}} \mu g_n$ ($\text{SUP}_{n \in \mathbb{N}} \mu g_n$ exists because of proposition (3, § 2.3, 26)). Then $\int g d\mu = \alpha = \int f d\mu$. Since $f \geq f_i \quad \forall i \in I, f \geq g_n \quad \forall n \in \mathbb{N}$, and hence $f \geq g$. Since $\int f d\mu = \int g d\mu$, it follows that $f = g$.

If $h \in L^\infty(X; \mathfrak{X}; \mu)$ is such that $h \geq f_i \forall i \in I$, then $h \geq g_n \forall n \in \mathbb{N}$ and since $g = f = \text{SUP}_\mu g_n$, it follows that $h \geq f$. Hence $f = \text{SUP}_\mu f_i$.

□

Corollary 29. *Let $(f_i)_{i \in I}$ be a directed increasing family of elements of $L^\infty(X; \mathfrak{X}; \mu)$ and let $f \in L^\infty(X; \mathfrak{X}; \mu)$. Then $f = \text{SUP}_{i \in I} f_i$ if and only if*

- (i) $f \geq f_i \forall i \in I$ and
- (ii) *there exists an increasing sequence $(f_n)_{n \in \mathbb{N}}$, $f_n, \forall n \in \mathbb{N}$, belonging to the family $(f_i)_{i \in I}$ such that $\int f_n d\mu \uparrow \int f d\mu$.*

Proof. This is an immediate consequence of the above proposition. □

Proposition 30. *Let ρ be a lifting from $L^\infty(X; \mathfrak{X}; \mu)$ to $\mathcal{B}(X; \mathfrak{X})$. Let $(f_i)_{i \in I}$ be a directed increasing family of elements of $L^\infty(X; \mathfrak{X}; \mu)$ bounded in norm. Let $f = \text{SUP}_{i \in I} f_i$. Then,*

$$\rho(f) \in \text{SUP}_{i \in I} \underline{\rho(f_i)} \text{ and } \forall_\mu x, \rho(f)(x) = \sup_{i \in I} \rho(f_i)(x).$$

In particular,

$$\sup_{i \in I} \rho(f_i) \in \hat{\mathfrak{X}}_\mu.$$

Proof. Note that $\rho(f) \in \text{SUP}_{i \in I} \underline{\rho(f_i)}$ is clear since $\text{SUP}_{i \in I} \underline{\rho(f_i)} = \text{SUP}_\mu f_i$ (since $\rho(f_i) \in f_i \forall i \in I$), $\rho(f) \in f$ and $f = \text{SUP}_{i \in I} f_i$. Since $\forall i \in I, f \geq f_i$, $\rho(f)(x) \geq \rho(f_i)(x)$ for all $x \in X$ and hence

$$\rho(f)(x) \geq \sup_{i \in I} \rho(f_i)(x)$$

for all $x \in X$.

Let $(f_n)_{n \in \mathbb{N}}$ be an increasing sequence of elements, $\forall n \in \mathbb{N}$, f_n belonging to the family $(f_i)_{i \in I}$ such that $f = \text{SUP}_{n \in \mathbb{N}} f_n$. (Such a sequence exists because of the proposition (3, § 2.3, 28)).

Then $\rho(f_n) \uparrow$ and $\int f_n d\mu \uparrow \int f d\mu$. Hence, $\int \rho(f_n) d\mu \uparrow \int \rho(f) d\mu$. Since $\rho(f_n) \uparrow$, by the monotone convergence theorem,

$$\sup_n \int \rho(f_n) d\mu = \int \sup_n \rho(f_n) d\mu.$$

Hence, $\int \rho(f) d\mu = \int \sup_n \rho(f_n) d\mu$. But $\sup_n \rho(f_n)(x) \leq \rho(f)(x)$ for all $x \in X$, since $\forall n \in \mathbb{N}$, $\rho(f_n)(x) \leq \rho(f)(x)$ for all $x \in X$. Hence $\forall_\mu x$, $\rho(f)(x) = \sup_n \rho(f_n)(x)$.

Now, for all $x \in X$.

$$\rho(f)(x) \geq \sup_{i \in I} \rho(f_i)(x) \geq \sup_n \rho(f_n)(x).$$

Hence, $\forall_\mu x$, $\rho(f)(x) = \sup_{i \in I} \rho(f_i)(x)$. Since $\rho(f) \in \hat{\mathfrak{X}}_\mu$ and since $\forall_\mu x$, $\sup_{i \in I} \rho(f_i)(x)$ is equal to $\rho(f)(x)$, it follows that

$$\sup_{i \in I} \rho(f_i) \in \hat{\mathfrak{X}}_\mu.$$

□

- 39 Let \mathfrak{z} be a sub σ -algebra of $\hat{\mathfrak{X}}_\mu$. Let $f \in L^\infty(X; \mathfrak{X}; \mu)$. Let g be any function on X , $g \in f$. Since $\forall_\mu x$, $g(x) \leq \|f\|_\infty$ and μ is a finite measure, it follows that g is μ -integrable, and hence $g^\mathfrak{z}$, a conditional expectation of g with respect to \mathfrak{z} exists. $\forall_\mu x$, $|g^\mathfrak{z}(x)| \leq \|f\|_\infty$ and hence $g^\mathfrak{z}$ is essentially bounded. Since any two conditional expectations of g with respect to \mathfrak{z} are equal μ -almost everywhere, we have a unique element of $L^\infty(X; \mathfrak{z}; \mu)$ to which any conditional expectation of g with respect to \mathfrak{z} belongs. Note that this element of $L^\infty(X; \mathfrak{z}; \mu)$ is independent of the function g , chosen to belong to f and hence depends only on f . We denote this element by $f^\mathfrak{z}$ and call it the *conditional expectation* of f with respect to \mathfrak{z} .

Proposition 31. *Let \mathfrak{z} be a sub σ -algebra of $\hat{\mathfrak{X}}_\mu$. Let $(f_i)_{i \in I}$ be a directed increasing family of elements of $L^\infty(X; \mathfrak{X}; \mu)$ bounded in norm. Let $f = \sup_{i \in I} f_i$. Then $\sup_{i \in I} f_i^\mathfrak{z}$ exists and is equal to $f^\mathfrak{z}$.*

Proof. Since $(f_i)_{i \in I}$ is a directed increasing family bounded in norm, $(f_i^3)_{i \in I}$ is also directed increasing and bounded in norm. Hence $\text{SUP}_{i \in I} \mu f_i^3$ exists. \square

To prove that $f^3 = \text{SUP}_{i \in I} \mu f_i^3$, it is sufficient to prove, because of proposition (3, § 2.3, 28), that $f^3 \geq f_i^3 \forall i \in I$ and $\int f^3 d\mu = \sup_{i \in I} \int f_i^3 d\mu$. But this follows immediately, since $f \geq f_i \forall i \in I$, $\int f d\mu = \int f^3 d\mu$, $\int f_i d\mu = \int f_i^3 d\mu \forall i \in I$, and $\int f d\mu = \sup_{i \in I} \int f_i d\mu$.

2.4 Radon Measures

Let X be a topological space. (By topological spaces in these Notes, we always mean only non-void, Hausdorff topological spaces). Let \mathfrak{X} be its Borel σ -algebra i.e. the σ -algebra generated by all the open sets of X .

Definition 32. A positive measure μ on \mathfrak{X} is said to be a Radon measure on X if

- (i) μ is locally finite i.e. every point $x \in X$ has a neighbourhood V_x such that $\mu(V_x) < +\infty$. and
- (ii) μ is inner regular in the sense that

$$\forall B \in \mathfrak{X}, \mu(B) = \sup \mu(K)$$

$$K \subset B$$

$$K \text{ compact}$$

Definition 33. Let μ be a Radon measure on a topological space X . Let $(K_i)_{i \in I}$ be a family of compact sets of X and N , a μ -null set. $\{(K_i)_{i \in I}, N\}$ is said to be a μ -concassage of X if

- (i) $X = N \cup \bigcup_{i \in I} K_i$
- (ii) $K_i \cap K_j = \emptyset$ if $i \neq j$, and

- (iii) the family $(K_i)_{i \in I}$ is locally countable in the sense that every point has a neighbourhood which has a non-void intersection with at most a countable number of K_i 's.

The following important theorem is stated here without proof; and for a proof see L. Schwartz [2], page 48, theorem 13.

Theorem. If μ is a Radon measure on topological space X , there exists a μ -concassage $\{(K_i)_{i \in I}, N\}$ of X .

Proposition 34. Let μ be a finite Radon measure on X . Then there exists a μ -concassage $\{(K_i)_{i \in I}, N\}$ of X with I countable.

Proof. Since μ is a finite Radon measure, using the inner regularity of μ , we see that there exists a μ -null set N_1 and a sequence $(X_n)_{n \in \mathbb{N}}$ of compact sets of X such that

$$X = N_1 \cup \bigcup_{n \in \mathbb{N}} X_n.$$

- 41 Let $\{(K_j)_{j \in J}, N_2\}$ be a μ -concassage of X . Since the family $(K_j)_{j \in J}$ is locally countable, every compact set can have a non-void intersection only with at most a countable number of $(K_j)_{j \in J}$. Hence, $\forall n \in \mathbb{N}$, there exists a countable set $I_n \subset J$ such that if $j \notin I_n$, $X_n \cap K_j = \emptyset$. Let $I = \bigcup_{n \in \mathbb{N}} I_n$. Then it is clear that $\{(K_i)_{i \in I}, N\}$ is a μ -concassage of X where $N = N_1 \cup N_2$. Note the I is countable. \square

3 Uniqueness Theorem

Let (X, \mathfrak{X}) be a measurable space. Let μ be a positive measure on \mathfrak{X} .

Definition 35. \mathfrak{X} is said to have the μ -countability property if there exists a set $N \in \mathfrak{X}$ with $\mu(N) = 0$ such that the σ -algebra $\mathfrak{X} \cap \mathcal{C}N$ on $X' = X \cap \mathcal{C}N$ consisting of sets of the form $A \cap \mathcal{C}N$, $A \in \mathfrak{X}$ is countably generated.

Examples. Suppose \mathfrak{X} is countably generated as in the case when \mathfrak{X} is the Borel σ -algebra of a topological space X having the 2nd axiom of

countability, then obviously \mathfrak{X} has the μ -countability property for any positive measure μ on \mathfrak{X} . Also, it can be easily proved that if $\hat{\mathfrak{X}}_\mu$ is countably generated, then \mathfrak{X} has the μ -countability property.

From now onwards, in this section, let $(\Omega, \mathcal{O}, \lambda)$ be a measure space and let (Y, \mathcal{Y}) be a measurable space. Let ν be a measure valued function on Ω with values in $\mathfrak{m}^+(Y, \mathcal{Y})$, $\nu \in \hat{\mathcal{O}}_\lambda$. Let \mathcal{C} be a σ -algebra on Ω , $\mathcal{C} \subset \hat{\mathcal{O}}_\lambda$. Let λ restricted to \mathcal{C} be σ -finite. Let $J = \int \nu_w d\lambda(w)$.

Theorem 36 (The Uniqueness theorem). *Let J be a σ -finite measure on \mathcal{Y} and let \mathcal{Y} have the J -countability property. Then, if $\nu_1^\mathcal{C}$ and $\nu_2^\mathcal{C}$ are any two conditional expectations of ν with respect to \mathcal{C} , we have $\forall \lambda w, (\nu_1^\mathcal{C})_w = (\nu_2^\mathcal{C})_w$ where $(\nu_i^\mathcal{C})_w$ for $i = 1, 2$ stands for the measure associated by $\nu_i^\mathcal{C}$ to w .*

Proof. First note that $J = \int \nu_1^\mathcal{C} d\lambda = \int \nu_2^\mathcal{C} d\lambda$. Since J is σ -finite, it therefore follows that $\forall \lambda w, (\nu_1^\mathcal{C})_w$ and $(\nu_2^\mathcal{C})_w$ are both σ -finite i.e. \exists a set $N_1 \in \hat{\mathcal{O}}_\lambda$ with $\lambda(N_1) = 0$ such that if $w \notin N_1$, $(\nu_1^\mathcal{C})_w$ and $(\nu_2^\mathcal{C})_w$ are σ -finite measures on \mathcal{Y} . 42

Since \mathcal{Y} has the J -countability property there exists a set $N \in \mathcal{Y}$ with $J(N) = 0$ such that the σ -algebra $\mathcal{Y}' = \mathcal{Y} \cap \mathcal{C}N$ on $Y' = Y \cap \mathcal{C}N$ is countably generated. Since $J(N) = 0$, it follows that $\forall \lambda w, (\nu_1^\mathcal{C})_w(N) = (\nu_2^\mathcal{C})_w(N) = 0$. i.e., there exists a set $N_2 \in \hat{\mathcal{O}}_\lambda$ with $\lambda(N_2) = 0$ such that if $w \notin N_2$, $(\nu_1^\mathcal{C})_w(N) = (\nu_2^\mathcal{C})_w(N) = 0$.

With our assumptions that J is σ -finite and \mathcal{Y}' is countably generated, we can find a class \mathcal{B} of subsets of Y' such that \mathcal{B} is countable, \mathcal{B} generates \mathcal{Y}' and $J(B) < +\infty \quad \forall B \in \mathcal{B}$. Let \mathcal{C} be the class of subsets of Y' formed by the sets which are finite intersections of sets belonging to \mathcal{B} . Then \mathcal{C} is countable, \mathcal{C} is a π -system generating \mathcal{Y}' and $J(C) < +\infty \quad \forall C \in \mathcal{C}$. There exists a set $N_3 \in \hat{\mathcal{O}}_\lambda$ with $\lambda(N_3) = 0$ such that if $w \notin N_3$, $(\nu_1^\mathcal{C})_w(C)$ and $(\nu_2^\mathcal{C})_w(C)$ are both finite for all $C \in \mathcal{C}$.

Now $\forall B \in \mathcal{Y}, \forall \lambda w, (\nu_1^\mathcal{C})_w(B) = (\nu_2^\mathcal{C})_w(B)$ since both $\nu_1^\mathcal{C}(B)$ and $\nu_2^\mathcal{C}(B)$ are conditional expectations with respect to \mathcal{C} of the extended real valued function $\nu(B)$. Hence in particular, $\forall C \in \mathcal{C}, \forall \lambda w, (\nu_1^\mathcal{C})_w(C) = (\nu_2^\mathcal{C})_w(C)$.

Since \mathcal{C} is countable, we therefore have,

$$\forall \lambda w, \forall C \in \mathcal{C}, (\nu_1^\mathcal{C})_w(C) = (\nu_2^\mathcal{C})_w(C) \text{ i.e.}$$

there exists a set $N_4 \in \hat{\mathcal{O}}_\lambda$ with $\lambda(N_4) = 0$ such that if $w \notin N_4$,

$$(\nu_1^\mathcal{C})_w(C) = (\nu_2^\mathcal{C})_w(C)$$

for all $C \in \mathcal{C}$.

Let $w \notin N_1 \cup N_3 \cup N_4$. Then both $(\nu_1^\mathcal{C})_w$ and $(\nu_2^\mathcal{C})_w$ are σ -finite, $(\nu_1^\mathcal{C})_w(C) = (\nu_2^\mathcal{C})_w(C) \quad \forall C \in \mathcal{C}$ and $(\nu_1^\mathcal{C})_w(C) < +\infty, (\nu_2^\mathcal{C})_w(C) < +\infty$ for all $C \in \mathcal{C}$. Since \mathcal{C} is a π -system generating \mathcal{Y}' and countable, by proposition (3, § 2.1, 24), we conclude that $(\nu_1^\mathcal{C})_w = (\nu_2^\mathcal{C})_w$ on \mathcal{Y}' . Hence if $w \notin N_1 \cup N_3 \cup N_4$, the measures $(\nu_1^\mathcal{C})_w$ and $(\nu_2^\mathcal{C})_w$ are equal on \mathcal{Y}' .

43 Therefore, if $w \notin N_1 \cup N_3 \cup N_4 \cup N_2$, the measures $(\nu_1^\mathcal{C})_w$ and $(\nu_2^\mathcal{C})_w$ are equal on \mathcal{Y} since if $w \notin N_2$, $(\nu_1^\mathcal{C})_w(N) = (\nu_2^\mathcal{C})_w(N) = 0$. Hence

$$\forall \lambda w, (\nu_1^\mathcal{C})_w = (\nu_2^\mathcal{C})_w \text{ on } \mathcal{Y}.$$

□

4 Existence theorems

Let X be a topological space and let \mathfrak{X} be its Borel σ -algebra. Let μ be a positive measure on \mathfrak{X} .

Definition 37. X is said to have the μ -compactness (resp. μ -compactness metrizable) property if there exists a set $N \in \mathfrak{X}$ with $\mu(N) = 0$ and a sequence $(K_n)_{n \in \mathbb{N}}$ of compact sets (resp. compact metrizable sets) such that $X = \bigcup_{n \in \mathbb{N}} K_n \cup N$ and $\forall n \in \mathbb{N}, \mu(K_n) < +\infty$.

Note that if μ is a σ -finite Radon measure on X , X has the μ -compactness property. If further to μ being a σ -finite Radon measure, either X is metrizable or every compact subset of X is metrizable, then X has the μ -compactness metrizable property. In particular, if X is a Suslin space and μ is a σ -finite Radon measure on X , then X has the μ -compactness metrizable property. (In a Suslin space, every compact subset is metrizable).

We shall now give some sufficient conditions for the existence of conditional expectations of measure valued functions.

Throughout this section, we shall adopt the following notations.

$(\Omega, \mathcal{O}, \lambda)$ is a measure space. Y is a topological space and \mathcal{Y} is its Borel- σ -algebra. ν is a measure valued function on Ω with values in $m^+(Y, \mathcal{Y})$, $\nu \in \hat{\mathcal{O}}_\lambda$ and having an integral J with respect to λ .

Theorem 38. *Let a sequence $(K_n)_{n \in \mathbb{N}}$ of compact sets of Y and a set $N \in \mathcal{Y}$ exist with the following properties:*

$$(i) \quad Y = \bigcup_{n \in \mathbb{N}} K_n \cup N$$

$$(ii) \quad J(N) = 0 \text{ and}$$

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$$(iii) \quad \forall n \in \mathbb{N}, \text{ the restriction of } J \text{ to } K_n \text{ is a Radon measure on } K_n.$$

Let \mathcal{C} be a σ -algebra contained in $\hat{\mathcal{O}}_\lambda$ such that \mathcal{C} is complete with respect to λ and let λ restricted to \mathcal{C} be σ -finite. Then, a conditional expectation of ν with respect to \mathcal{C} exists.

Proof. Let us split the proof in two cases case 1 and case 2. In case 1, we assume that Y is compact and J is a Radon measure on Y . In this case, the assumptions (i), (ii) and (iii) mentioned in the statement of the theorem are trivially verified. In case 2, we shall consider a general topological space Y , having the properties (i), (ii) and (iii) mentioned in the statement of the theorem. We shall deduce case 2 from case 1.

The proof in case 1 proceeds in three steps, Step 1, Step 2 and Step 3.

In Step 1, we define a measure valued function $\nu^\mathcal{C}$ on Ω with values in $m^+(Y, \mathcal{Y})$ such that $\forall w \in \Omega$, $\nu_w^\mathcal{C}$ is a Radon measure on Y and such that \forall real valued continuous function φ on Y , $w \rightarrow \nu_w^\mathcal{C}(\varphi)$ is a conditional expectation with respect to \mathcal{C} of the function $w \rightarrow \nu_w(\varphi)$.

In Step 2, we prove that $\forall U$ open in Y , $\nu^\mathcal{C}(\chi_U)$ is a conditional expectation with respect to \mathcal{C} of the function $\nu(\chi_U)$.

In Step 3, we prove that $\forall B \in \mathcal{Y}$, $\nu^\mathcal{C}(\chi_B)$ is a conditional expectation with respect to \mathcal{C} of the function $\nu(\chi_B)$.

Case 1. Y a compact space and J , a Radon measure on Y .

Step 1. Since J is a Radon measure on Y , $J(Y) < +\infty$. Hence $\forall \lambda w$, $\nu_w(Y) < +\infty$. Without loss of generality, we can assume that $\forall w \in \Omega$, $\nu_w(Y) < +\infty$. For, consider the measure valued function ν' on Ω with values in $m^+(Y, \mathcal{Y})$ given by

$$\nu'_w = \begin{cases} \nu_w & \text{if } \nu_w(Y) < +\infty \\ 0 & \text{otherwise i.e. the zero measure if } \nu_w(Y) = +\infty. \end{cases}$$

45 Then ν' is a measure valued function on Ω with values in $m^+(Y, \mathcal{Y})$, $\nu' \in \hat{\mathcal{O}}_\lambda$, ν' has the same integral J as ν and further $\forall w \in \Omega$, ν'_w is a finite measure. Also, $\forall \lambda w$, $\nu'_w = \nu_w$. Hence, if a conditional expectation of ν' with respect to \mathcal{C} exists, it is a conditional expectation of ν with respect to \mathcal{C} , as well.

Hence, we may assume that $\forall w \in \Omega$, $\nu_w(Y) < +\infty$.

Let φ be any real valued bounded function on Y , $\varphi \in \mathcal{Y}$. Since $\forall w \in \Omega$, ν_w is a finite measure on \mathcal{Y} , $\forall w \in \Omega$, φ is ν_w -integrable. Consider the real valued function $\nu(\varphi)$ taking w to $\nu_w(\varphi)$. Since λ restricted to \mathcal{C} is σ -finite, a conditional expectation for $\nu(\varphi)$ with respect to \mathcal{C} exists. Let $[\nu(\varphi)]^\mathcal{C}$ be a conditional expectation of $\nu(\varphi)$ with respect to \mathcal{C} .

Now, let us fix $[\nu(1)]^\mathcal{C}$ once and for all as *the* conditional expectation of $\nu(1)$ with respect to \mathcal{C} in such a way that $\forall w \in \Omega$, $0 \leq [\nu(1)]^\mathcal{C}(w) < +\infty$.

Let $\|\varphi\|$ be $\sup_{y \in Y} |\varphi(y)|$

Then, $|\nu(\varphi)| \leq \|\varphi\| \cdot \nu(1)$.

Hence, $\forall \lambda w$, $|[\nu(\varphi)]^\mathcal{C}|(w) \leq |\nu(\varphi)|^\mathcal{C}(w) \leq \|\varphi\|[\nu(1)]^\mathcal{C}(w)$. Hence, if $A = \{w : [\nu(1)]^\mathcal{C}(w) = 0\}$, then $A \in \mathcal{C}$ and $\forall \lambda w$, $w \in A$, $[\nu(\varphi)]^\mathcal{C}(w) = 0$.

Define the quotient $\frac{[\nu(\varphi)]^\mathcal{C}}{[\nu(1)]^\mathcal{C}}$ to be zero on the set A . On $\complement A$, the quotient $\frac{[\nu(\varphi)]^\mathcal{C}}{[\nu(1)]^\mathcal{C}}$ has a meaning.

Now, $\forall \lambda w$, $\left| \frac{[\nu(\varphi)]^\mathcal{C}}{[\nu(1)]^\mathcal{C}}(w) \right| \leq \|\varphi\|$. Hence the function $\frac{[\nu(\varphi)]^\mathcal{C}}{[\nu(1)]^\mathcal{C}}$ is es-

entially bounded. Since it belongs to \mathcal{C} ,

$$\frac{[\nu(\varphi)]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}} \in L^{\infty}(\Omega; \mathcal{C}; \lambda).$$

Note that this element $\frac{[\nu(\varphi)]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}}$ of $L^{\infty}(\Omega; \mathcal{C}; \lambda)$ is independent of the choice of the conditional expectation $[\nu(\varphi)]^{\mathcal{C}}$ of $\nu(\varphi)$ and hence depends only on φ .

Let ρ be a lifting from $L^{\infty}(\Omega; \mathcal{C}; \lambda)$ to $\mathcal{B}(\Omega; \mathcal{C})$. The existence of a lifting is guaranteed by the lifting theorem mentioned in § 2.2 of this chapter. 46

Let $\mathcal{C}(Y)$ be the space of all real valued continuous functions on Y .

If $w \in \Omega$, define the maps $\nu_w^{\mathcal{C}}$ on $\mathcal{C}(Y)$ taking real value as follows.

If $\psi \in \mathcal{C}(Y)$, define $\nu_w^{\mathcal{C}}(\psi)$ as $[\nu(1)]^{\mathcal{C}}(w) \cdot \rho\left(\frac{[\nu(\psi)]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}}\right)(w)$.

Then, by the properties of the lifting $\nu_w^{\mathcal{C}}$ is a positive linear functional on $\mathcal{C}(Y)$ and hence defines a positive Radon measure on Y . Let $\nu^{\mathcal{C}}$ be the measure valued function on Ω with values in $\mathfrak{m}^+(Y, \mathcal{Y})$ taking w to $\nu_w^{\mathcal{C}}$.

Since \mathcal{C} is complete, $\forall \psi \in \mathcal{C}(Y)$, $\rho\left(\frac{[\nu(\psi)]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}}\right) \in \mathcal{C}$ and hence $\forall \psi \in \mathcal{C}(Y)$, the function $\nu^{\mathcal{C}}(\psi)$ belongs to \mathcal{C} . Moreover, if $B \in \mathcal{C}$,

$$\begin{aligned} \int_B \nu_w^{\mathcal{C}}(\psi) d\lambda(w) &= \int_B [\nu(1)]^{\mathcal{C}}(w) \cdot \rho\left(\frac{[\nu(\psi)]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}}\right)(w) d\lambda(w) \\ &= \int_B [\nu(1)]^{\mathcal{C}} \cdot \frac{[\nu(\psi)]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}} d\lambda \\ &= \int_B [\nu(\psi)]^{\mathcal{C}} d\lambda \\ &= \int_B \nu_w^{\mathcal{C}}(\psi) d\lambda(w). \end{aligned}$$

Hence $\nu^{\mathcal{C}}(\psi)$ is a conditional expectation with respect to \mathcal{C} of the function $\nu(\psi)$.

Step 2. Let U be an open subset of Y , $U \neq \emptyset$. Since U is open, χ_U is a lower-semi-continuous function on Y and hence there exists a directed increasing family $(\varphi_i)_{i \in I}$ of continuous functions on Y such that $\forall i \in I$, $0 \leq \varphi_i \leq \chi_U$ and $\sup_{i \in I} \varphi_i = \chi_U$.

$$\forall i \in I, \forall \lambda w, \left| \frac{[\nu(\varphi_i)]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}}(w) \right| \leq \|\varphi_i\| \leq 1.$$

Hence, the directed increasing family $\frac{[\nu(\varphi_i)]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}}$ of elements of $L^\infty(\Omega; \mathcal{C}; \lambda)$ is bounded in norm and hence $(\text{SUP})_{i \in I} \lambda \frac{[\nu(\varphi_i)]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}}$ exists by proposition (3, § 2.3, 27).

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We claim that

$$\text{SUP}_{i \in I} \lambda \frac{[\nu(\varphi_i)]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}} = \frac{[\nu(\chi_U)]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}}$$

To prove this, it is sufficient to prove because of Corollary (3, § 2.3, 29) that

$$\frac{[\nu(\chi_U)]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}} \geq \frac{[\nu(\varphi_i)]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}} \forall i \in I,$$

and that there exists an increasing sequence $(\varphi_n)_{n \in \mathbb{N}}$ of continuous functions, $\varphi_n \forall n \in \mathbb{N}$, belonging to the family $(\varphi_i)_{i \in I}$ such that

$$\lim_{n \rightarrow \infty} \int \frac{[\nu(\varphi_n)]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}} d\lambda = \int \frac{[\nu(\chi_U)]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}} d\lambda$$

Since $\forall i \in I$, $\varphi_i \leq \chi_U$, we have

$$\forall i \in I, \nu(\varphi_i) \leq \nu'(\chi_U).$$

Hence $\forall i \in I, \forall \lambda w, [\nu(\varphi_i)]^{\mathcal{C}}(w) \leq [\nu(\chi_U)]^{\mathcal{C}}(w)$. Therefore,

$$\forall i \in I, \frac{[\nu(\varphi_i)]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}} \leq \frac{[\nu(\chi_U)]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}}.$$

Since J is a Radon measure, and $(\varphi_i)_{i \in I}$ is a directed increasing family of continuous functions with $\chi_U = \sup_{i \in I} \varphi_i$, we have

$$J(\chi_U) = \sup_{i \in I} J(\varphi_i).$$

Hence there exists an increasing sequence $(\varphi_n)_{n \in \mathbb{N}}$ of continuous functions, $\forall n \in \mathbb{N}$, φ_n belonging to the family $(\varphi_i)_{i \in I}$ such that

$$\forall y, \varphi_n(y) \uparrow \chi_U(y).$$

Hence,

$$\forall \lambda w, \forall y, \varphi_n(y) \uparrow \chi_U(y),$$

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Therefore,

$$\forall \lambda w, \nu_w(\varphi_n) \uparrow \nu_w(\chi_U).$$

Since $w \rightarrow [\gamma(1)]^{\mathcal{C}}(w) \in \mathcal{C}$ and $A \in \mathcal{C}$, by property (iv) of the conditional expectations of extended real valued functions mentioned in § 5 of Chapter 1 (actually the property of $\nu_{\mathcal{C}, \mathcal{C}}$ mentioned there is used), we have

$$\begin{aligned} \forall n \in \mathbb{N}, \int \frac{[\nu(\varphi_n)]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}}(w) d\lambda(w) &= \int \frac{[\nu\varphi_n]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}}(w) d\lambda(w) \\ &= \int_A \frac{\nu_w(\varphi_n)}{[\nu(1)]^{\mathcal{C}}(w)} d\lambda(w) \end{aligned}$$

Hence $\forall n \in \mathbb{N}$, $\int \frac{[\nu(\varphi_n)]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}} d\lambda = \int_A \frac{\nu_w(\varphi_n)}{[\nu(1)]^{\mathcal{C}}(w)} d\lambda(w)$. Since $\forall \lambda w$, $\nu_w(\varphi_n) \uparrow \nu_w(\chi_U)$,

$$\int_A \frac{\nu_w(\varphi_n)}{[\nu(1)]^{\mathcal{C}}(w)} d\lambda(w) \uparrow \int_A \frac{\nu_w(\chi_U)}{[\nu(1)]^{\mathcal{C}}(w)} d\lambda(w).$$

Again by the same property for $\nu_{\mathcal{C}, \mathcal{C}}$ mentioned in (iv) of the conditional expectations of extended real valued functions mentioned in § 5 of Chapter 1,

$$\int_A \frac{\nu_w(\chi_U)}{[\nu(1)]^{\mathcal{C}}(w)} d\lambda(w) = \int_A \frac{[\nu(\chi_U)]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}}(w) d\lambda(w)$$

$$\begin{aligned}
&= \int \frac{[\nu(\chi_U)]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}}(w) d\lambda(w) \\
&= \int \frac{[\nu(\chi_U)]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}} d\lambda
\end{aligned}$$

Hence, $\lim_{n \rightarrow \infty} \int \frac{[\nu(\varphi_n)]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}} d\lambda$ exists and is equal to $\int \frac{[\nu(\chi_U)]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}} d\lambda$. This proves that $\frac{[\nu(\chi_U)]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}} = \text{SUP}_{i \in I} \frac{[\nu(\varphi_i)]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}}$ and therefore, by proposition (3, § 2.3, 30),

$$\forall \lambda w, \rho \left(\frac{[\nu(\chi_U)]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}} \right) (w) = \sup_{i \in I} \rho \left(\frac{[\nu(\varphi_i)]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}} \right) (w).$$

49 and $\sup_{i \in I} \rho \left(\frac{[\nu(\varphi_i)]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}} \right) \in \mathcal{C}$ since \mathcal{C} is complete. Since $\forall w \in \Omega$, $\nu_w^{\mathcal{C}}$ is a Radon measure,

$$\begin{aligned}
\nu_w^{\mathcal{C}}(\chi_U) &= \sup_{i \in I} \nu_w^{\mathcal{C}}(\varphi_i) \\
&= \sup_{i \in I} [\nu(1)]^{\mathcal{C}}(w) \rho \left(\frac{[\nu(\varphi_i)]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}} \right) (w) \\
&= [\nu(1)]^{\mathcal{C}}(w) \cdot \sup_{i \in I} \rho \left(\frac{[\nu(\varphi_i)]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}} \right) (w).
\end{aligned}$$

Hence $\nu^{\mathcal{C}}(\chi_U) \in \mathcal{C}$, since both $[\nu(1)]^{\mathcal{C}}$ and $\sup_{i \in I} \rho \left(\frac{[\nu(\varphi_i)]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}} \right)$ belong to \mathcal{C} .

Let $B \in \mathcal{C}$. Then,

$$\begin{aligned}
\int_B \nu_w^{\mathcal{C}}(\chi_U) d\lambda(w) &= \int_B [\nu(1)]^{\mathcal{C}}(w) \sup_{i \in I} \rho \left(\frac{[\nu(\varphi_i)]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}} \right) (w) d\lambda(w) \\
&= \int_B [\nu(1)]^{\mathcal{C}}(w) \frac{[\nu(\chi_U)]^{\mathcal{C}}}{[\nu(1)]^{\mathcal{C}}}(w) d\lambda(w)
\end{aligned}$$

$$\begin{aligned}
&= \int_B [v(\chi_U)]^{\mathcal{C}}(w) d\lambda(w) \\
&= \int_B v_w(\chi_U) d\lambda(w).
\end{aligned}$$

Hence $v^{\mathcal{C}}(\chi_U)$ is a conditional expectation of $v(\chi_U)$.

Step 3. Let us prove that $\forall B \in \mathcal{Y}$, the function $v^{\mathcal{C}}(\chi_B)$ belongs to \mathcal{C} and is a conditional expectation with respect to \mathcal{C} of the function $v(\chi_B)$.

Consider the class \mathcal{C} of all sets $B \in \mathcal{Y}$ for which $v^{\mathcal{C}}(\chi_B) \in \mathcal{C}$ and is a conditional expectation with respect to \mathcal{C} of the function $v(B)$. It is easily seen that \mathcal{C} is a d -system.

From Step 2, \mathcal{C} contains the class \mathcal{U} of all open sets of Y , \mathcal{U} is a π -system generating \mathcal{Y} . Hence, by the Monotone class theorem, \mathcal{C} contains \mathcal{Y} and hence $\mathcal{C} = \mathcal{Y}$. 50

Hence $\forall B \in \mathcal{Y}$, the function $v^{\mathcal{C}}(\chi_B)$ belongs to \mathcal{C} and is a conditional expectation of the function $v(\chi_B)$ with respect to \mathcal{C} . This shows that the measure valued function $v^{\mathcal{C}}$ on Ω with values in $m^+(Y, \mathcal{Y})$ is a conditional expectation of v with respect to \mathcal{C} .

Thus, Case 1 is completely proved.

Case 2. Let Y be an arbitrary topological space having the properties stated in the theorem, i.e. \exists a sequence $(K_n)_{n \in \mathbb{N}}$ of compact sets of Y and a set $N \in \mathcal{Y}$ with $J(N) = 0$ such that $Y = \bigcup_{n \in \mathbb{N}} K_n \cup N$ and such that the measure J_{K_n} , the restriction of J to K_n is a Radon measure on K_n . Let

$$X_n = K_n \setminus (K_1 \cup \dots \cup K_{n-1}).$$

Then $\forall n \in \mathbb{N}$, X_n is a Borel set of Y , $X_n \cap X_m = \emptyset$ if $n \neq m$, $Y = \bigcup_{n \in \mathbb{N}} X_n \cup N$ and the measure J_{X_n} , the restriction of J to X_n is a finite Radon measure on X_n . By proposition (3, § 2.4, 34), $\forall n \in \mathbb{N}$, \exists a sequence $(X_n^m)_{m \in \mathbb{N}}$ of mutually disjoint compact sets of X_n and a Borel set N_n of X_n with $J_{X_n}(N_n) = 0$ such that $\{(X_n^m)_{m \in \mathbb{N}}, N_n\}$ is a J_{X_n} -concassage of X_n .

Thus, there exists a sequence $(Y_n)_{n \in \mathbb{N}}$ of compact sets of Y and a set $M \in \mathcal{Y}$ with $J(M) = 0$ such that $Y = \bigcup_{n \in \mathbb{N}} Y_n \cup M$, $Y_n \cap Y_m = \emptyset$ if $n \neq m$ and the measure J_{Y_n} , the restriction of J to Y_n is a Radon measure on Y_n . Let $\forall n$, \mathcal{Y}_n be the Borel σ -algebra of Y_n .

Let $\forall n \in \mathbb{N}$, ν^n be the measure valued function on Ω with values in $m^+(Y_n, \mathcal{Y}_n)$, taking $w \in \Omega$ to the measure ν_w^n which is the restriction of ν_w to Y_n . Then, $\forall n \in \mathbb{N}$, $\nu^n \in \hat{\mathcal{O}}_\lambda$. $\forall B \in \mathcal{Y}_n$,

$$\int \nu_w^n(B) d\lambda(w) = \int \nu_w(B) d\lambda(w) = J(B) = J_{Y_n}(B).$$

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Hence the integral of ν^n with respect to λ is the measure J_{Y_n} which is a Radon measure on Y_n . Hence, by Case 1, $\forall n \in \mathbb{N}$, a conditional expectation $\nu^{\mathcal{C}, n}$ of ν^n with respect to \mathcal{C} exists.

Define $\forall n \in \mathbb{N}$, $\forall w \in \Omega$, the measures $\nu_w^{\mathcal{C}, n}$ on \mathcal{Y} as follows:

If $B \in \mathcal{Y}$, define $\nu_w^{\mathcal{C}, n}(B)$ as equal to $\nu_w^{\mathcal{C}, n}(B \cap Y_n)$.

Then $\forall n \in \mathbb{N}$, $\forall w \in \Omega$, $\nu_w^{\mathcal{C}, n}$ is a positive measure on \mathcal{Y} and the measure valued function $\nu^{\mathcal{C}, n}$ on Ω with values in $m^+(Y, \mathcal{Y})$ taking w to $\nu_w^{\mathcal{C}, n}$ belongs to \mathcal{C} .

Define $\forall w \in \Omega$, the measure $\nu_w^{\mathcal{C}}$ on \mathcal{Y} as $\nu_w^{\mathcal{C}} = \sum_{n \in \mathbb{N}} \nu_w^{\mathcal{C}, n}$ i.e. $\forall B \in \mathcal{Y}$, $\nu_w^{\mathcal{C}}(B) = \sum_{n \in \mathbb{N}} \nu_w^{\mathcal{C}, n}(B)$. Then, the measure valued function $\nu^{\mathcal{C}}$ on Ω with values in $m^+(Y, \mathcal{Y})$ taking w to the measure $\nu_w^{\mathcal{C}}$ belongs to \mathcal{C} .

Let $C \in \mathcal{C}$ and $B \in \mathcal{Y}$. Then,

$$\begin{aligned} \int_C \nu_w^{\mathcal{C}}(B) d\lambda(w) &= \int_C \left(\sum_{n \in \mathbb{N}} \nu_w^{\mathcal{C}, n}(B) \right) d\lambda(w) \\ &= \sum_{n \in \mathbb{N}} \int_C \nu_w^{\mathcal{C}, n}(B) d\lambda(w) \\ &= \sum_{n \in \mathbb{N}} \int_C \nu_w^{\mathcal{C}, n}(B \cap Y_n) d\lambda(w) \\ &= \sum_{n \in \mathbb{N}} \int_C \nu_w^n(B \cap Y_n) d\lambda(w) \end{aligned}$$

since $\forall n \in \mathbb{N}$, $\nu^{\mathcal{C},n}(B \cap Y_n)$ is a conditional expectation with respect to \mathcal{C} of $\nu^n(B \cap Y_n)$. So the left side above is

$$\begin{aligned} &= \sum_{n \in \mathbb{N}} \int_C \nu_w(B \cap Y_n) d\lambda(w) \\ &= \int_C \left(\sum_{n \in \mathbb{N}} \nu_w(B \cap Y_n) \right) d\lambda(w). \end{aligned}$$

Since $J(M) = 0$, $\forall \lambda w$, $\nu_w(M) = 0$. Hence, $\forall B \in \mathcal{B}$, $\forall \lambda w$, $\sum_{n \in \mathbb{N}} \nu_w(B \cap Y_n) = \nu_w(B)$ since the sequence $(Y_n)_{n \in \mathbb{N}}$ is mutually disjoint and $Y = \bigcup_{n \in \mathbb{N}} Y_n \cup M$. 52

Hence,

$$\int_C \sum_{n \in \mathbb{N}} \nu_w(B \cap Y_n) d\lambda(w) = \int_C \nu_w(B) d\lambda(w).$$

Therefore, $\nu^{\mathcal{C}}(B)$ is a conditional expectation with respect to \mathcal{C} of the function $\nu(B) \forall B \in \mathcal{B}$. Hence $\nu^{\mathcal{C}}$ is a conditional expectation of ν with respect to \mathcal{C} . \square

Remark 39. The assumptions in the above theorem regarding Y and J are fulfilled if J is a σ -finite Radon measure on Y and these are stronger than the J -compactness property for Y .

Lemma 40. Let Y be a compact metrizable space. Let \mathcal{C} be a σ -algebra contained in $\hat{\mathcal{C}}_\lambda$ and let $\hat{\mathcal{C}}_\lambda$ be the completion of \mathcal{C} with respect to λ . Let $\nu^{\hat{\mathcal{C}}_\lambda}$ be a measure valued function on Ω with values in $m^+(Y, \mathcal{B})$ belonging to $\hat{\mathcal{C}}_\lambda$ such that $\forall w \in \Omega$, $\nu_w^{\hat{\mathcal{C}}_\lambda}$ is a Radon measure on Y and J , the integral of $\nu^{\hat{\mathcal{C}}_\lambda}$ with respect to λ be a finite measure on \mathcal{B} . Then, there exists a measure valued function $\nu^{\mathcal{C}}$ on Ω with values in $m^+(Y, \mathcal{B})$ belonging to \mathcal{C} , such that $\forall w \in \Omega$, $\nu_w^{\mathcal{C}}$ is a Radon measure on Y and $\forall \lambda w$, $\nu_w^{\mathcal{C}} = \nu_w^{\hat{\mathcal{C}}_\lambda}$.

Proof. Since Y is a compact metrizable space, the Banach space $\mathcal{C}(Y)$ of all real valued continuous functions on Y has a countable dense set

D . If $\psi \in \mathcal{C}(Y)$, let $\|\psi\|$ be $\sup_{y \in Y} |\psi(y)|$. We can assume that D has the following property, namely, given any $f \in \mathcal{C}(Y)$, $f \geq 0$ and any $\epsilon > 0$, there exists a function $\varphi_\epsilon \in D$, $\varphi_\epsilon \geq 0$ such that $\|f - \varphi_\epsilon\| \leq \epsilon$. i.e., the positive elements of $\mathcal{C}(Y)$ can be approximated at will by positive elements of D . Let a sequence $(\varphi_n)_{n \in \mathbb{N}}$ of continuous functions on Y , constitute the set D .

Now, $\forall n \in \mathbb{N}$, the function $v_w^{\mathcal{C}_\lambda}(\varphi_n)$ belongs to \mathcal{C}_λ and is λ -integrable. Hence $\forall n \in \mathbb{N}$, there exist functions f_1^n and f_2^n on Ω belonging to \mathcal{C} such that

$$\forall w \in \Omega, f_1^n(w) \leq v_w^{\mathcal{C}_\lambda}(\varphi_n) \leq f_2^n(w)$$

53 and the set $B_n = \{w \in \Omega \mid f_1^n(w) \neq f_2^n(w)\}$ has λ -measure zero.

Let $B = \bigcup_{n \in \mathbb{N}} B_n$. Then $B \in \mathcal{C}$ and $\lambda(B) = 0$.

Let $\varphi \in \mathcal{C}(Y)$. There exists a sequence φ_{n_k} of functions belonging to D such that

$$\|\varphi_{n_k} - \varphi\| \rightarrow 0 \text{ as } n_k \rightarrow \infty.$$

Since $\forall w, v_w^{\mathcal{C}_\lambda}$ is a Radon measure on Y and hence $\forall w, v_w^{\mathcal{C}_\lambda}(Y) < +\infty$ since Y is compact, we have $\forall w \in \Omega, v_w^{\mathcal{C}_\lambda}(\varphi_{n_k}) \rightarrow v_w^{\mathcal{C}_\lambda}(\varphi)$ as $n_k \rightarrow \infty$.

Hence, if $w \notin B$, $\lim_{n_k \rightarrow \infty} f_1^{n_k}(w)$ and $\lim_{n_k \rightarrow \infty} f_2^{n_k}(w)$ exist and both are equal to $\lim_{n_k \rightarrow \infty} v_w^{\mathcal{C}_\lambda}(\varphi_{n_k})$ which is $v_w^{\mathcal{C}_\lambda}(\varphi)$.

Therefore, $\forall \varphi \in \mathcal{C}(Y)$, $\forall w \notin B$, the $\lim_{n_k \rightarrow \infty} f_1^{n_k}(w)$ is independent of the choice of the sequence (φ_{n_k}) chosen to converge to φ in $\mathcal{C}(Y)$.

Hence, $\forall w \in \Omega$, define the map $v_w^{\mathcal{C}}$ on $\mathcal{C}(Y)$ as,

$$v_w^{\mathcal{C}}(\varphi) = \begin{cases} \lim_{n_k \rightarrow \infty} f_1^{n_k}(w), & \text{if } w \notin B \\ 0 & \text{if } w \in B. \end{cases}$$

where $\varphi \in \mathcal{C}(Y)$ and $(\varphi_{n_k})_{n_k \in \mathbb{N}}$ is a sequence such that $\forall n_k \in \mathbb{N}, \varphi_{n_k} \in D$ and $\varphi_{n_k} \rightarrow \varphi$ as $n_k \rightarrow \infty$ in $\mathcal{C}(Y)$. $\forall w \in \Omega$, the map $v_w^{\mathcal{C}}$ defined on $\mathcal{C}(Y)$ as above is clearly linear and if $w \notin B, v_w^{\mathcal{C}}(\varphi) = v_w^{\mathcal{C}_\lambda}(\varphi)$ for all $\varphi \in \mathcal{C}(Y)$. i.e. $\forall w, \forall \varphi \in \mathcal{C}(Y), \gamma_w^{\mathcal{C}}(\varphi) = v_w^{\mathcal{C}_\lambda}(\varphi)$. Moreover the linear functional $v_w^{\mathcal{C}}$ is positive because of our assumptions on D that the positive elements of

$\mathcal{C}(Y)$, can be approximated by positive elements of D . Hence, $\forall w \in \Omega$, $\nu_w^{\mathcal{C}}$ is a Radon measure on Y .

Since $\forall \lambda w, \forall \varphi \in \mathcal{C}(\varphi) = \nu_w^{\mathcal{C}_\lambda}(\varphi)$ and since $\forall w \in \Omega$, $\nu_w^{\mathcal{C}}$ and $\nu_w^{\mathcal{C}_\lambda}$ are Radon measures, it follows that $\forall \lambda w$, the measures $\nu_w^{\mathcal{C}}$ and $\nu_w^{\mathcal{C}_\lambda}$ are equal, i.e.

$$\forall \lambda w, \nu_w^{\mathcal{C}} = \nu_w^{\mathcal{C}_\lambda}.$$

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We have to prove that the measure valued function $\nu^{\mathcal{C}}$ on Ω taking w to $\nu_w^{\mathcal{C}}$ belongs to \mathcal{C} .

If $\varphi \in \mathcal{C}(Y)$, it is clear that the function $\nu^{\mathcal{C}}(\varphi)$ belongs to \mathcal{C} since $\forall n \in \mathbb{N}, f_1^n \in \mathcal{C}$ and $B \in \mathcal{C}$.

Let U be a non-void open set. Since Y is metrizable, there exists an increasing sequence φ_n of continuous functions on Y such that $\forall n \in \mathbb{N}, 0 \leq \varphi_n \leq \chi_U$ and $\varphi_n(y) \uparrow \chi_U(y)$ for all $y \in Y$.

Hence $\forall w \in \Omega, \nu_w^{\mathcal{C}}(\chi_U) = \lim_{n \rightarrow \infty} \nu_w^{\mathcal{C}}(\varphi_n)$. Therefore, $\forall U$ open, $\nu^{\mathcal{C}}(\chi_U) \in \mathcal{C}$.

Now, a standard application of the Monotone class theorem will yield that $\forall C \in \mathcal{Y}$, the function $\nu^{\mathcal{C}}(\chi_C)$ belongs to \mathcal{C} .

Hence the measure valued function $\nu^{\mathcal{C}}$ belongs to \mathcal{C} and since $\forall \lambda w, \nu_w^{\mathcal{C}} = \nu_w^{\mathcal{C}_\lambda}$, our lemma is completely proved. \square

Theorem 41. *Let Y have the J -compactness metrizability property. Let \mathcal{C} be a σ -algebra contained in $\hat{\mathcal{C}}_\lambda$ and let λ restricted to \mathcal{C} be σ -finite. Then a conditional expectation of ν with respect to \mathcal{C} exists and is unique.*

Proof. Let us prove the theorem under the assumption that Y is a compact metrizable space and J is a finite measure on \mathcal{Y} . The general case will follow along lines similar to case 2 of theorem (3, § 4, 38).

Since J is a finite measure on \mathcal{Y} and Y is a compact metrizable space, J is a Radon measure on Y . Let $\hat{\mathcal{C}}_\lambda$ be the completion of \mathcal{C} with respect to λ . By case 1 of theorem (3, § 4, 38), a conditional expectation $\nu^{\hat{\mathcal{C}}_\lambda}$ of ν with respect to $\hat{\mathcal{C}}_\lambda$ exists in such a way that $\forall w \in \Omega, \nu_w^{\hat{\mathcal{C}}_\lambda}$ is a Radon measure on Y .

$$\int \nu_w^{\hat{\mathcal{C}}_\lambda}(Y) d\lambda(w) = \int \nu_w(Y) d\lambda(w) = J(Y) < +\infty.$$

55 Hence, by the above lemma (3, § 4, 40), there exists a measure valued function $\nu^{\mathcal{C}}$ on Ω with values in $m^+(Y, \mathcal{Y})$ such that $\nu^{\mathcal{C}} \in \mathcal{C}$ and $\forall_{\lambda} w$, $\nu_w^{\mathcal{C}} = \nu_w^{\mathcal{C}_{\lambda}}$.

Hence, since $\nu^{\mathcal{C}_{\lambda}}$ is a conditional expectation of ν with respect to $\hat{\mathcal{C}}_{\lambda}$, $\nu^{\mathcal{C}}$ is a conditional expectation of ν with respect to \mathcal{C} .

Since the Borel σ -algebra of a compact metrizable space is countably generated, we can easily see that if Y has the J -compactness metrizable property, \mathcal{Y} has the J -countability property. Hence by theorem (3, § 3, 36) the conditional expectation of ν with respect to \mathcal{C} is unique. \square

5 Existence theorem for disintegration of a measure. The theorem of M. Jirina

Throughout this section, let Ω be a topological space, \mathcal{O} its Borel σ -algebra and λ a positive measure on \mathcal{O} .

We have already remarked in § 1 of this chapter, that a disintegration of λ with respect to \mathcal{C} is a conditional expectation of the measure valued function δ on Ω with values in $m^+(\Omega, \mathcal{O})$ taking w to the Dirac measure δ_w and vice versa. So, by theorem (3, § 3, 36) we get the following uniqueness theorem for disintegrations and by the theorems (3, § 4, 38) and (3, § 4, 41), we get the following two theorems for the existence of disintegrations. More precisely, we have

Theorem 42 (Uniqueness). *Let λ be a σ -finite measure on \mathcal{O} and let \mathcal{O} have the λ -countability property. Let \mathcal{C} be a σ -algebra contained in $\hat{\mathcal{C}}_{\lambda}$ and let λ restricted to \mathcal{C} be σ -finite. Then, if $\{(\lambda_1^{\mathcal{C}})_w\}_{w \in \Omega}$ and $\{(\lambda_2^{\mathcal{C}})_w\}_{w \in \Omega}$ are two disintegrations of λ with respect to \mathcal{C} , we have $\forall_{\lambda} w$, $(\lambda_1^{\mathcal{C}})_w = (\lambda_2^{\mathcal{C}})_w$.*

Theorem 43 (Existence(M.Jirina)). *Let there exist a sequence $(K_n)_{n \in \mathbb{N}}$ of compact sets of \mathcal{O} and a set $N \in \mathcal{O}$ such that*

56 (i) $\Omega = \bigcup_{n \in \mathbb{N}} K_n \cup N$,

(ii) $\lambda(N) = 0$, and

(iii) $\forall n \in \mathbb{N}$, the restriction of λ to K_n is a Radon measure on K_n .

Let \mathcal{C} be a complete σ -algebra contained in $\hat{\mathcal{O}}_\lambda$ i.e. let $\mathcal{C} = \hat{\mathcal{C}}_\lambda$. Then, a disintegration of λ with respect to \mathcal{C} exists.

Theorem 44 (Existence). *Let Ω have the λ -compactness metrizable property. Let \mathcal{C} be a σ -algebra contained in $\hat{\mathcal{O}}_\lambda$ and let $\lambda|_{\mathcal{C}}$ be σ -finite, where $\lambda|_{\mathcal{C}}$ stands for the restriction on λ to \mathcal{C} . Then, a disintegration of λ with respect to \mathcal{C} exists and is unique.*

We remark that the assumptions in the theorem (3, § 5, 43) are fulfilled if λ is a σ -finite Radon measure on Ω . Ω has the λ -compactness metrizable property if λ is a σ -finite Radon measure on Ω and if Ω is either metrizable or if every compact subset of Ω is metrizable. In particular, if Ω is a Suslin space and if λ is a σ -finite Radon measure on Ω , then Ω has the λ -compactness metrizable property.

When λ is a Radon probability measure on Ω i.e., λ is a Radon measure and $\lambda(\Omega) = 1$, the theorem (3, § 5, 43) is essentially due to M. Jirina [1] in the sense that this theorem is an easy consequence of his theorem 3.2, on page 448 and the ‘note added in proof’ in page 450.

6 Another kind of existence theorem for conditional expectation of measure valued functions

Throughout this section, let Ω be a topological space, \mathcal{O} its Borel σ -algebra, λ a positive measure on \mathcal{O} , Y a topological space and \mathcal{Y} its Borel σ -algebra.

If \mathcal{C} is a σ -algebra of $\hat{\mathcal{O}}_\lambda$, we saw in Chapter 1 how the existence of a disintegration of λ with respect to \mathcal{C} , implies immediately the existence of conditional expectation with respect to \mathcal{C} of non-negative, extended real valued functions on Ω belonging to \mathcal{O} and in Chapter 2, we saw how the existence of a disintegration of λ with respect to \mathcal{C} implies the existence of conditional expectation with respect to $\hat{\mathcal{C}}_\lambda$ of extended real valued λ -integrable functions belonging to $\hat{\mathcal{O}}_\lambda$ and also of Banach

space valued λ -integrable functions belonging to $\hat{\mathcal{O}}_\lambda$. In this section, we shall consider the case of measure valued functions in relation to their conditional expectations with respect to \mathcal{C} , when a disintegration of λ with respect to \mathcal{C} exists.

Theorem 45. *Let \mathcal{C} be a σ -algebra contained in $\hat{\mathcal{O}}_\lambda$. Let $(\lambda_w^\mathcal{C})_{w \in \Omega}$ be a disintegration of λ with respect to \mathcal{C} . Let ν be a measure valued function on Ω with values in $\mathfrak{m}^+(Y, \mathcal{Y})$. Let $\underline{\nu} \in \mathcal{O}$. Then the measure valued function $\nu^\mathcal{C}$ on Ω with values in $\mathfrak{m}^+(Y, \mathcal{Y})$ defined as $\nu_w^\mathcal{C} = \int \nu_{w'} \lambda_w^\mathcal{C}(dw')$ is a conditional expectation of ν with respect to \mathcal{C} . In particular, a conditional expectation of ν with respect to \mathcal{C} exists.*

Proof. Let f be a function on Y , $f \geq 0$, $f \in \mathcal{Y}$. Consider the extended real valued function $\nu(f)$. This function belongs to \mathcal{O} since $\nu \in \mathcal{O}$. Since $(\lambda_w^\mathcal{C})_{w \in \Omega}$ is a disintegration of λ with respect to \mathcal{C} , the function $w \rightarrow \int \nu_{w'}(f) \lambda_w^\mathcal{C}(dw')$ is a conditional expectation of the function $\nu(f)$ with respect to \mathcal{C} . Hence the measure valued function $\nu^\mathcal{C}$ on Ω with values in $\mathfrak{m}^+(Y, \mathcal{Y})$ defined as $\nu_w^\mathcal{C} = \int \nu_{w'} \lambda_w^\mathcal{C}(dw') \forall w \in \Omega$, is a conditional expectation of ν with respect to \mathcal{C} . \square

When $\nu \in \hat{\mathcal{O}}_\lambda$, we cannot apply the above argument since for $f \geq 0$ on Y , belonging to \mathcal{Y} , $\nu(f)$, though belongs to $\hat{\mathcal{O}}_\lambda$, does not in general belong to \mathcal{O} . Hence, the integral $\int \nu_{w'}(f) \lambda_w^\mathcal{C}(dw')$ does not have a meaning in general for all functions f on Y , $f \geq 0$, $f \in \mathcal{Y}$, since the measures $\lambda_w^\mathcal{C}$ are measures on \mathcal{O} and not on $\hat{\mathcal{O}}_\lambda$ for all $w \in \Omega$.

58 However, for functions ν belonging to $\hat{\mathcal{O}}_\lambda$, we have the following theorem of existence of conditional expectations.

Theorem 46. *Let \mathcal{C} be a σ -algebra contained in $\hat{\mathcal{O}}_\lambda$. Let $(\lambda_w^\mathcal{C})_{w \in \Omega}$ be a disintegration of λ with respect to \mathcal{C} . Let ν be a measure valued function on Ω with values in $\mathfrak{m}^+(Y, \mathcal{Y})$, ν belonging to $\hat{\mathcal{O}}_\lambda$. Let $J = \int \nu_w d\lambda(w)$. Let J be σ -finite and let \mathcal{Y} have the J -countability property. Then, a conditional expectation of ν with respect to \mathcal{C} exists and is unique.*

For the proof of this theorem, we need the following lemma.

Lemma 47. *Let ν be a measure valued function on Ω with values in $\mathfrak{m}^+(Y, \mathcal{Y})$, ν belonging to $\hat{\mathcal{O}}_\lambda$. Let $J = \int \nu_w d\lambda(w)$. Let J be σ -finite. Let*

\mathcal{Y} have the J -countability property. Then there exists a measure valued function ν' on Ω with values in $m^+(Y, \mathcal{Y})$ such that $\nu' \in \mathcal{O}$ and $\forall \lambda w, \nu'_w = \nu_w$ on \mathcal{Y} .

Proof. Since J is σ -finite, there exists an increasing sequence $(E_n)_{n \in \mathbb{N}}$ of sets belonging to \mathcal{Y} such that $Y = \bigcup_{n \in \mathbb{N}} E_n$ and $J(E_n) < +\infty \forall n \in \mathbb{N}$.

Hence,

$$\forall n \in \mathbb{N}, \quad \forall \lambda w, \nu_w(E_n) < +\infty,$$

Hence,

$$\forall \lambda w, \forall n \in \mathbb{N}, \nu_w(E_n) < +\infty.$$

i.e. \exists a set $N_1 \in \mathcal{O}$ with $\lambda(N_1) = 0$ such that if $w \notin N_1, \nu_w(E_n) < +\infty$ for all $n \in \mathbb{N}$.

Let $N \in \mathcal{Y}$ with $J(N) = 0$ such that the σ -algebra $\mathcal{Y} \cap \mathcal{C}N$ on $Y' = Y \cap \mathcal{C}N$ is countably generated.

Since $J(N) = 0, \exists N_2 \in \mathcal{O}$ with $\lambda(N_2) = 0$ such that if $w \notin N_2, \nu_w(N) = 0$.

Let $\mathcal{C} = (C_n)_{n \in \mathbb{N}}, \forall n \in \mathbb{N}, C_n \in \mathcal{Y}'$ generate \mathcal{Y}' . We can assume that \mathcal{C} is a π -system and that $Y' \in \mathcal{C}$.

Consider $\forall n \in \mathbb{N}, \forall m \in \mathbb{N}$, the function $w \rightarrow \nu_w(E_m \cap C_n)$. This function belongs to $\hat{\mathcal{O}}_\lambda$ and $E_m \cap C_n \in \mathcal{Y}$. Therefore, there exist functions $f_1^{m,n}$ and $f_2^{m,n}$ on Ω belonging to \mathcal{O} such that $f_1^{m,n}(w) \leq \nu_w(E_m \cap C_n) \leq f_2^{m,n}(w)$ for all $w \in \Omega$ and the set

$$N^{m,n} = \{w \in \Omega \mid f_1^{m,n}(w) \neq f_2^{m,n}(w)\}$$

has λ -measure zero.

Let $N_3 = \bigcup_{\substack{m \in \mathbb{N} \\ n \in \mathbb{N}}} N^{m,n}$. Then $N_3 \in \mathcal{O}$ and $\lambda(N_3) = 0$.

If y_0 is any point of Y , define the measure valued function ν' on Ω with values in $m^+(Y, \mathcal{Y})$ as follows.

$$\nu'_w = \begin{cases} \nu_w, & \text{if } w \notin N_1 \cup N_2 \cup N_3 \\ \delta_{y_0}, & \text{if } w \in N_1 \cup N_2 \cup N_3 \end{cases}$$

From the definition, it is clear that $\forall \lambda w, \nu'_w = \nu_w$. To prove the lemma, we have to prove only that $\nu' \in \mathcal{O}$. Now,

$\forall m \in \mathbb{N}, \forall n \in \mathcal{N}, \forall w \in \Omega,$

$$\begin{aligned} v'_w(E_m \cap C_n) &= \chi_{C_{\{N_1 \cup N_2 \cup N_3\}}}(w) \cdot v_w(E_m \cap C_n) + \chi_{N_1 \cup N_2 \cup N_3}(w) \cdot \chi_{E_m \cap C_n}(y_0) \\ &= \chi_{C_{\{N_1 \cup N_2 \cup N_3\}}}(w) \cdot f_1^{m,n}(w) + \chi_{N_1 \cup N_2 \cup N_3}(w) \cdot \chi_{E_m \cap C_n}(y_0). \end{aligned}$$

Since $f_1^{m,n} \in \mathcal{O}$ and $N_1 \cup N_2 \cup N_3 \in \mathcal{O}$, it is clear that the function $v'(E_m \cap C_n)$ belongs to \mathcal{O} .

Now fix a $m \in \mathbb{N}$. Let

$$\mathcal{B} = \{B \in \mathcal{Y}' \mid v'(B \cap E_m) \text{ belongs to } \mathcal{O}\}.$$

It is clear that \mathcal{B} is a d -system. \mathcal{B} contains the π -system \mathcal{C} which generates \mathcal{Y}' . Hence, by the Monotone class theorem, $\mathcal{B} = \mathcal{Y}'$. Hence, since m is arbitrary, $\forall m \in \mathbb{N}, \forall B \in \mathcal{Y}', v'(B \cap E_m)$ belongs to \mathcal{O} . Since $\forall B \in \mathcal{Y}', \forall w \in \Omega, v'_w(B) = \lim_{m \rightarrow \infty} v'_w(B \cap E_m)$, it follows that $v'(B)$ belongs to \mathcal{O} .

Let

$$A \in \mathcal{Y}, A = A \cap Y' \cup A \cap N.$$

60 If $w \notin N_1 \cup N_2 \cup N_3, v'_w(A) = v'_w(A \cap Y')$, since $v'_w(A \cap N) = v_w(A \cap N) = 0$. Hence,

$$v'_w(A) = \chi_{C_{\{N_1 \cup N_2 \cup N_3\}}}(w) \cdot v'_w(A \cap Y') + \chi_{N_1 \cup N_2 \cup N_3}(w) \cdot \chi_A(y_0)$$

for all $w \in \Omega$. Since $A \cap Y' \in \mathcal{Y}'$, the function $v'(A \cap Y')$ belongs to \mathcal{O} . Since $N_1 \cup N_2 \cup N_3 \in \mathcal{O}$, it follows from the above expression of $v'(A)$, that $v'(A)$ belongs to \mathcal{O} .

Since A is an arbitrary set $\in \mathcal{Y}$, it follows that the measure valued function v' belongs to \mathcal{O} . \square

Proof of theorem 46 From the above lemma (3, § 6, 47) there exists a measure valued function v' on Ω with values in $m^+(Y, \mathcal{Y})$ such that $v' \in \mathcal{O}$ and $\forall_{\lambda} w, v'_w = v_w$. From theorem (3, § 6, 45) there exists a conditional expectation $v'^{\mathcal{C}}$ with respect to \mathcal{C} for v' , since $v' \in \mathcal{O}$. Since $\forall_{\lambda} w, v'_w = v_w$, it is clear that $v'^{\mathcal{C}}$ is also a conditional expectation of v with respect to \mathcal{C} .

The uniqueness follows from theorem (3, § 3, 36).

Thus, we see, if a disintegration of λ with respect to \mathcal{C} exists, how theorem (3, § 6, 45) guarantees immediately the existence of conditional expectations with respect to \mathcal{C} of measure valued functions belonging to \mathcal{O} and how theorem (3, § 6, 46) guarantees the same for measure valued functions belonging to $\hat{\mathcal{O}}_\lambda$ if J is σ -finite and if \mathcal{Y} has the J -countability property where J is the integral of ν with respect to λ . But theorems (3, § 5, 43) and (3, § 5, 44) give sufficient conditions for the existence of disintegration of λ . Hence, we have the following existence theorem of conditional expectations for measure valued functions.

Theorem 48. (i) Let \mathcal{C} be a complete σ -algebra contained in $\hat{\mathcal{O}}_\lambda$ and let there exist a sequence $(K_n)_{n \in \mathbb{N}}$ of compact sets of Ω and a set $N \in \mathcal{O}$ with $\lambda(N) = 0$ such that $\Omega = \bigcup_{n \in \mathbb{N}} K_n \cup N$ and the restriction of λ to $K_n \forall n \in \mathbb{N}$ is a Radon measure on K_n .

Or (ii) Let \mathcal{C} be an arbitrary σ -algebra contained in $\hat{\mathcal{O}}_\lambda$ and let Ω have the λ -compactness metrizable property. 61

Then, under either of the conditions (i) and (ii), if ν is any measure valued function on Ω with values in $m^+(Y, \mathcal{Y})$, $\nu \in \mathcal{O}$, a conditional expectation of ν with respect to \mathcal{C} exists.

If $\nu \in \hat{\mathcal{O}}_\lambda$, if $J = \int \nu_w d\lambda(w)$, if J is σ -finite, and if \mathcal{Y} has the J -countability property, then under either of the conditions (i) and (ii), a conditional expectation of ν with respect to \mathcal{C} exists and is unique.

7 Conditions for a given family $(\lambda_w^\mathcal{C})_{w \in \Omega}$ of positive measures on \mathcal{O} , to be a disintegration of λ with respect to \mathcal{C} and consequences

Throughout this section, let $(\Omega, \mathcal{O}, \lambda)$ be a measure space and let \mathcal{C} be a σ -algebra contained in $\hat{\mathcal{O}}_\lambda$. Let $(\lambda_w^\mathcal{C})_{w \in \Omega}$ be a given family of positive measures on \mathcal{O} . We shall discuss below some necessary and sufficient conditions for this family to be a disintegration of λ , with respect to \mathcal{C} .

Proposition 49. For $(\lambda_w^\mathcal{C})_{w \in \Omega}$ to be a disintegration of λ , it is necessary and sufficient that the following three conditions are verified.

- (i) The measure valued function $\lambda^\mathcal{C}$ taking $w \in \Omega$ to $\lambda_w^\mathcal{C}$, belongs to \mathcal{C}
- (ii) $\lambda = \int \lambda_w^\mathcal{C} d\lambda(w)$ and
- (iii) $\forall A \in \mathcal{C}, \forall \lambda w, \lambda_w^\mathcal{C}$ is carried by A or by $\complement A$ according as $w \in A$ or $w \in \complement A$.

Proof. Necessity. Let $(\lambda_w^\mathcal{C})_{w \in \Omega}$ be a disintegration of λ with respect to \mathcal{C} . Then, by the definition of disintegration, (i) and (ii) are fulfilled. Let us verify (iii).

Let $A \in \mathcal{C}$. $\lambda^\mathcal{C}(\complement A)$ is a conditional expectation of $\chi_{\complement A}$ with respect to \mathcal{C} . Therefore,

$$\int_A \chi_{\complement A}(w) d\lambda(w) = \int_A \lambda_w^\mathcal{C}(\complement A) d\lambda(w)$$

i.e. $0 = \lambda(A \cap \complement A) = \int_A \lambda_A^\mathcal{C}(\complement A) d\lambda(w).$

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Therefore, $\forall \lambda w, \chi_A(w). \lambda_A^\mathcal{C}(\complement A) = 0$. Similarly,

$$\forall \lambda w, \chi_{\complement A}(w). \lambda_w^\mathcal{C}(A) = 0.$$

This proves that $\forall \lambda w, \lambda_w^\mathcal{C}(\complement A) = 0$ if $w \in A$ and $\lambda_w^\mathcal{C}(A) = 0$, if $w \in \complement A$. Hence, $\forall \lambda w, \lambda_w^\mathcal{C}$ is carried by A if $w \in A$ and is carried by $\complement A$ if $w \in \complement A$.

Sufficiency. We have to prove that $\forall B \in \mathcal{O}$. $\lambda^\mathcal{C}(B)$ is a conditional expectation of χ_B with respect to \mathcal{C} . i.e., we have to prove that $\forall A \in \mathcal{C}$,

$$\int_A \chi_B d\lambda = \int_A \lambda^\mathcal{C}(B) d\lambda;$$

i.e. we have to prove that

$$\lambda(A \cap B) = \int_A \lambda^\mathcal{C}(B) d\lambda.$$

$$\int_A \lambda_w^{\mathcal{C}}(B) d\lambda(w) = \int_A \lambda_w^{\mathcal{C}}(B \cap A) d\lambda(w) + \int_{\mathcal{C}A} \lambda_w^{\mathcal{C}}(B \cap \mathcal{C}A) d\lambda(w).$$

Since $\forall_{\lambda w}, w \in A, \lambda_w^{\mathcal{C}}$ is carried by A ,

$$\forall_{\lambda w}, w \in A, \lambda_w^{\mathcal{C}}(B \cap \mathcal{C}A) = 0.$$

Hence,

$$\int_A \lambda_w^{\mathcal{C}}(B \cap \mathcal{C}A) d\lambda(w) = 0.$$

Therefore,

$$\int_A \lambda_w^{\mathcal{C}}(B) d\lambda(w) = \int_A \lambda_w^{\mathcal{C}}(A \cap B) d\lambda(w).$$

Since

$$\begin{aligned} \lambda &= \int \lambda_w^{\mathcal{C}} d\lambda(w), \\ \lambda(A \cap B) &= \int \lambda_w^{\mathcal{C}}(A \cap B) d\lambda(w) \\ &= \int_A \lambda_w^{\mathcal{C}}(A \cap B) d\lambda(w) + \int_{\mathcal{C}A} \lambda_w^{\mathcal{C}}(A \cap B) d\lambda(w). \end{aligned}$$

Since $\forall_{\lambda w}, \lambda_w^{\mathcal{C}}$ is carried by $\mathcal{C}A$ if $w \in \mathcal{C}A$,

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$$\int_{\mathcal{C}A} \lambda_w^{\mathcal{C}}(A \cap B) d\lambda(w) = 0.$$

Hence

$$\begin{aligned} \lambda(A \cap B) &= \int_A \lambda_w^{\mathcal{C}}(A \cap B) d\lambda(w) \\ &= \int_A \lambda_w^{\mathcal{C}}(B) d\lambda(w) \end{aligned}$$

□

Definition 50. Let (X, \mathfrak{X}) be a measurable space. Let $x \in X$. The \mathfrak{X} -atom of x is defined to be the intersection of all sets belonging to \mathfrak{X} and containing x .

Definition 51. Let (X, \mathfrak{X}) be a measurable space. We say \mathfrak{X} is countably separating if there exists a sequence $(A_n)_{n \in \mathbb{N}}$, $A_n \in \mathfrak{X} \forall n \in \mathbb{N}$, such that $\forall x \in X$, the \mathfrak{X} -atom of x is the intersection of all the A_n 's that contain x .

It can be easily proved that if \mathfrak{X} is countably generated, then it is countably separating.

Proposition 52. Let (i) the measure valued function $\lambda^{\mathcal{C}}$ on Ω taking w to $\lambda_w^{\mathcal{C}}$ belong to \mathcal{C} .

$$(i) \lambda = \int \lambda_w^{\mathcal{C}} d\lambda(w) \text{ and}$$

$$(ii) \forall \lambda w, \lambda_w^{\mathcal{C}} \text{ is carried by the } \mathcal{C}\text{-atom of } w.$$

Then $(\lambda_w^{\mathcal{C}})_{w \in \Omega}$ is a disintegration of λ .

Proof. Let $(\lambda_w^{\mathcal{C}})_{w \in \Omega}$ have (i), (ii) and (iii). From condition (iii), we see by the definition of the \mathcal{C} -atom of w , given $A \in \mathcal{C}$, $\forall \lambda w$, $\lambda_w^{\mathcal{C}}$ is carried by A if $w \in A$ and $\lambda_w^{\mathcal{C}}$ is carried by $\complement A$ if $w \in \complement A$. Thus, the condition (iii) of this proposition implies the condition (iii) of the proposition (3, § 7, 49). Hence, the conditions (i), (ii) and (iii) of this proposition imply the conditions (i), (ii) and (iii) of the proposition (3, § 7, 49). Hence $(\lambda_w^{\mathcal{C}})_{w \in \Omega}$ is a disintegration of λ with respect to \mathcal{C} \square

64 Proposition 53. Let \mathcal{C} be countably separating. Then, the conditions (i), (ii) and (iii) of the previous proposition (3, § 7, 52) are necessary for $(\lambda_w^{\mathcal{C}})_{w \in \Omega}$ to be a disintegration of λ with respect to \mathcal{C} .

Proof. Let \mathcal{C} be countably separating and let $(\lambda_w^{\mathcal{C}})_{w \in \Omega}$ be a disintegration of λ with respect to \mathcal{C} . Then (i) and (ii) are obvious. We have to only verify that $\forall \lambda w$, $\lambda_w^{\mathcal{C}}$ is carried by the \mathcal{C} -atom of w .

Since \mathcal{C} is countably separating, by definition, there exists a sequence $(A_n)_{n \in \mathbb{N}}$ of sets belonging to \mathcal{C} such that $\forall w \in \Omega$, the \mathcal{C} -atom of w is the intersection of all the A_n 's that contain w .

By condition (iii) of the proposition (3, § 7, 49), $\forall n \in \mathbb{N}, \exists$ a set $N_n \in \mathcal{O}$ with $\lambda(N_n) = 0$ such that if $w \notin N_n$, $\lambda_w^\mathcal{C}$ is carried by A_n if $w \in A_n$ and is carried by $\complement A_n$ if $w \notin A_n$.

Let $N = \bigcup_{n \in \mathbb{N}} N_n$. Then $N \in \mathcal{O}$ and $\lambda(N) = 0$. Let $w \notin N$ and let A_w be the \mathcal{C} -atom of w . Since \mathcal{C} is countably separating, there exists a sequence (n_k) of natural numbers such that

$$A_w = \bigcup_{A_{n_k} \ni w} A_{n_k},$$

A_{n_k} belonging to the sequence $(A_n)_{n \in \mathbb{N}} \forall n_k$.

For every n_k , since $w \in A_{n_k}$, $\lambda_w^\mathcal{C}$ is carried by A_{n_k} and hence is carried by A_w since $A_w = \bigcap_{A_{n_k} \ni w} A_{n_k}$.

Since $w \notin N$ is arbitrary, it follows that $\forall \lambda w$, $\lambda_w^\mathcal{C}$ is carried by the \mathcal{C} -atom of w . \square

The following *counter example* will show that the condition (iii) of the proposition (3, § 7, 52), namely $\forall \lambda w$, $\lambda_w^\mathcal{C}$ is carried by the \mathcal{C} -atom of w is not necessarily true for a disintegration $(\lambda_w^\mathcal{C})_{w \in \Omega}$ of λ with respect to \mathcal{C} , without further assumptions on \mathcal{C} .

Let Ω be the circle S^1 in \mathbb{R}^2 . Let \mathcal{O} be the Borel σ -algebra of Ω . Let λ be the Lebesgue measure of S^1 on \mathcal{O} . Let \mathcal{C} be the σ -algebra of all symmetric Borel sets. Since Ω is a compact metric space and λ is a Radon probability measure, by theorem (3, § 5, 44), a disintegration $(\lambda_w^\mathcal{C})_{w \in \Omega}$ of λ with respect to \mathcal{C} exists and is unique. Consider the family $(\delta_w^\mathcal{C})_{w \in \Omega}$ of measures given by $\delta_w^\mathcal{C} = \frac{1}{2}(\delta_w + \delta_{-w})$. It is easy to check that $(\delta_w^\mathcal{C})_{w \in \Omega}$ is a disintegration of λ with respect to \mathcal{C} . Since the disintegration is unique, we have $\forall \lambda w$, $\lambda_w^\mathcal{C} = \delta_w^\mathcal{C}$, i.e. $\forall \lambda w$, $\lambda_w^\mathcal{C} = \frac{1}{2}(\delta_w + \delta_{-w})$. 65

Let $\hat{\mathcal{C}}$ be the σ -algebra generated by \mathcal{C} and all the λ -null sets of \mathcal{O} . i.e. let $\hat{\mathcal{C}} = \mathcal{C} \vee \eta_\lambda$. By theorem (3, § 5, 44), a disintegration $(\lambda_w^{\hat{\mathcal{C}}})_{w \in \Omega}$ of λ with respect to $\hat{\mathcal{C}}$ exists and is unique. It is clear that $(\lambda_w^\mathcal{C})_{w \in \Omega}$ is also a disintegration of λ with respect to $\hat{\mathcal{C}}$. Hence, by uniqueness,

$$\forall \lambda w, \lambda_w^{\hat{\mathcal{C}}} = \lambda_w^\mathcal{C} = \frac{1}{2}(\delta_w + \delta_{-w}).$$

If $w \in \Omega$, the \mathcal{C} -atom of w is the single point w itself. Since we see that $\forall \lambda w, \lambda_w^{\mathcal{C}} = \frac{1}{2}(\delta_w + \delta_{-w})$, $\forall \lambda w, \lambda_w^{\mathcal{C}}$ is carried by the pair of points w and $-w$ and hence not by the \mathcal{C} -atom of w for these w for which $\lambda_w^{\mathcal{C}} = \frac{1}{2}(\delta_w + \delta_{-w})$. Hence, the condition (iii) of proposition (3, § 7, 52) is violated.

Let us now state and prove some simple, but useful consequences of the previous propositions.

Let (Z, \mathfrak{z}) be a measurable space such that \mathfrak{z} is countably separating and let $\forall z \in Z$, the \mathfrak{z} -atom of z be z itself. This means that there exists a sequence $(Z_n)_{n \in \mathbb{N}}$ of sets belonging to \mathfrak{z} such that every point $z \in Z$ is the intersection of a suitable subsequence of these Z_n 's.

Proposition 54. *Let h be a mapping from Ω to Z such that $h \in \mathcal{C}$. Let $(\lambda_w^{\mathcal{C}})_{w \in \Omega}$ be a disintegration of λ with respect to \mathcal{C} . Then,*

$$\forall \lambda w, \forall \lambda_w^{\mathcal{C}} w', h(w') = h(w).$$

66 *i.e. $\forall \lambda w, h$ is $\lambda_w^{\mathcal{C}}$ almost everywhere is a constant equal to $h(w)$.*

Proof. Let $w \in \Omega$. There exists a subsequence $(Z_{n_k})_{n_k \in \mathbb{N}}$ of $(Z_n)_{n \in \mathbb{N}}$ such that

$$h(w) = \bigcap_{n_k=1}^{\infty} Z_{n_k}.$$

Hence,

$$h^{-1}(h(w)) = \bigcap_{n_k=1}^{\infty} h^{-1}(Z_{n_k}).$$

$\forall n \in \mathbb{N}, h^{-1}(Z_n) \in \mathcal{C}$ since $h \in \mathcal{C}$.

Now, by condition (iii) of the proposition (3, § 7, 49), $\forall n \in \mathbb{N}, \forall \lambda w, \lambda_w^{\mathcal{C}}$ is carried by $h^{-1}(Z_n)$ if $w \in h^{-1}(Z_n)$. Hence, $\forall \lambda w, \forall n \in \mathbb{N}, \lambda_w^{\mathcal{C}}$ is carried by $h^{-1}(Z_n)$ if $w \in h^{-1}(Z_n)$.

Therefore, $\forall \lambda w, \lambda_w^{\mathcal{C}}$ is carried by the intersection of all those $h^{-1}(Z_n)$ for which $w \in h^{-1}(Z_n)$. But the intersection of all the $h^{-1}(Z_n)$ for which $w \in h^{-1}(Z_n)$ is $h^{-1}(h(w))$.

Hence, $\forall \lambda w, \lambda_w^{\mathcal{C}}$ is carried by $h^{-1}(h(w))$. This precisely means that

$$\forall \lambda w, \forall \lambda_w^{\mathcal{C}} w', h(w') = h(w).$$

□

Corollary 55. *Let $(h_n)_{n \in \mathbb{N}}$ be a sequence of functions on Ω with values in Z such that $\forall n, h_n \in \mathcal{C}$. Then,*

$$\forall \lambda w, \forall \lambda_w^{\mathcal{C}} w', \forall n \in \mathbb{N}, h_n(w') = h_n(w).$$

Proof. This is an immediate consequence of the above proposition (3, § 7, 54). □

Corollary 56. *Let \mathcal{O} be countably generated. Then $\forall \lambda w, \forall \lambda_w^{\mathcal{C}} w', \lambda_{w'}^{\mathcal{C}} = \lambda_w^{\mathcal{C}}$.*

Proof. Let $(B_n)_{n \in \mathbb{N}}$ be a π -system generating \mathcal{O} . Define a sequence $(h_n)_{n \in \mathbb{N}}$ of functions on Ω as $h_n(w) = \lambda_w^{\mathcal{C}}(B_n) \forall w \in \Omega$.

Then, $\forall n, h_n$ is a function on Ω with values in the extended real numbers and $h_n \in \mathcal{C} \forall n \in \mathbb{N}$. Hence, by the above Corollary (3, § 7, 55),

$$\forall \lambda w, \forall \lambda_w^{\mathcal{C}} w', \forall n \in \mathbb{N}, \lambda_{w'}^{\mathcal{C}}(B_n) = \lambda_w^{\mathcal{C}}(B_n).$$

A standard application of the Monotone class theorem therefore gives,

$$\forall \lambda w, \forall \lambda_w^{\mathcal{C}} w', \forall B \in \mathcal{O}, \lambda_{w'}^{\mathcal{C}}(B) = \lambda_w^{\mathcal{C}}(B).$$

Hence,

$$\forall \lambda w, \forall \lambda_w^{\mathcal{C}} w', \lambda_{w'}^{\mathcal{C}} = \lambda_w^{\mathcal{C}}.$$

□

Part II

Measure valued Supermartingales and Regular Disintegration of measures with respect to a family of σ -Algebras

Chapter 4

Supermartingales

1 Extended real valued Supermartingales

Throughout this chapter, let $(\Omega, \mathcal{O}, \lambda)$ be a measure space. Let $(\mathcal{C}^t)_{t \in \mathbb{R}}$ be a family of sub σ -algebras of $\hat{\mathcal{O}}_\lambda$, which is *increasing* in the sense that $\forall s, t \in \mathbb{R}, s \leq t, \mathcal{C}^s \subset \mathcal{C}^t$. Let us further assume that λ restricted to \mathcal{C}^t is σ -finite $\forall t \in \mathbb{R}$. If \mathcal{C} is a σ -algebra contained in $\hat{\mathcal{O}}_\lambda$ and f is a function on Ω with values in $\bar{\mathbb{R}}^+$, belonging to $\hat{\mathcal{O}}_\lambda$, $f^{\mathcal{C}}$ will stand for a conditional expectation of f with respect to \mathcal{C} . 69

If f is a function from $\Omega \times \mathbb{R}$ to $\bar{\mathbb{R}}^+$, f^t will denote $\forall t \in \mathbb{R}$, the function on Ω with values in $\bar{\mathbb{R}}^+$, taking $w \in \Omega$ to $f(w, t)$. $\forall w \in \Omega$, f_w will denote the function on \mathbb{R} to $\bar{\mathbb{R}}^+$, taking t to $f(w, t)$.

Let f be a function from $\Omega \times \mathbb{R}$ to $\bar{\mathbb{R}}^+$. We have the following series of definitions.

Definition 57. f is said to be adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$ if $\forall t, f^t \in \mathcal{C}^t$.

Definition 58. f is said to be right continuous if $\forall \lambda w, f_w$ is a right continuous function.

Definition 59. f is said to be regulated if $\forall \lambda w, f_w$ has finite right and finite left limits at all points $t \in \mathbb{R}$.

Definition 60. A function g from $\Omega \times \mathbb{R}$ to $\bar{\mathbb{R}}^+$ is said to be a modification or a version of f if $\forall t \in \mathbb{R}, \forall \lambda w, g(w, t) = f(w, t)$.

Definition 61. f is said to be a supermartingale (resp. martingale) adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$ if

- 70 (i) f is adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$ and
(ii) $\forall s, t \in \mathbb{R}, s \leq t, \forall_{\lambda} w, (f^t)^{\mathcal{C}^s}(w) \leq f^s(w)$.

(resp.

- (i) f is adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$ and
(ii) $\forall s, t \in \mathbb{R}, s \leq t, \forall_{\lambda} w, (f^t)^{\mathcal{C}^s}(w) = f^s(w)$)

Note that every martingale is a supermartingale. A simple example of a supermartingale adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$ is given by any function f from $\Omega \times \mathbb{R}$ to $\bar{\mathbb{R}}^+$, adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$ for which $\forall_{\lambda} w, f_w$ is a decreasing function of t .

Let g be any function on Ω with values in $\bar{\mathbb{R}}^+$ and belonging to $\hat{\mathcal{C}}_{\lambda}$. Let $\forall t, w \rightarrow f(w, t)$ be a given conditional expectation of g with respect to \mathcal{C}^t . Then, it is clear that the function f on $\Omega \times \mathbb{R}$ to $\bar{\mathbb{R}}^+$, taking (w, t) to $f(w, t)$ is a martingale, adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$.

If f is a supermartingale adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$, note that $t \rightarrow J^t = \int f^t d\lambda$ is a decreasing function of t . If f is a right continuous supermartingale, we easily see by applying Fatou's lemma that $t \rightarrow J^t$ is right continuous.

Definition 62. A supermartingale adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$, is said to be regular if it is right continuous and regulated.

Note that if f is a regular supermartingale adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$, then $\forall_{\lambda} w, f_w$ is a real valued function on \mathbb{R} .

Definition 63. A supermartingale g adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$ is said to be a regular modification of a supermartingale f adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$ if g is regular and is a modification of f .

Remark 64. If f is a supermartingale adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$ and if g_1 and g_2 are two regular modifications of f , we easily see that $\forall_{\lambda} w, \forall t, g_1(w, t) =$

71 $g_2(w, t)$. In this sense, we say that a regular modification of a supermartingale adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$ is unique if exists. In particular, if f is a regular supermartingale adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$ and if g is a regular modification of f , then $\forall \lambda w, \forall t, g(w, t) = f(w, t)$.

The following theorem is very fundamental in the theory of supermartingales. It guarantees the existence of a regular modification for a supermartingale, under some conditions. Whenever, we refer to the following theorem, we refer to it as ‘the fundamental theorem’.

The Fundamental theorem. Let f be a supermartingale adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$ such that $\forall t \in \mathbb{R}, J^t < +\infty$ and $t \rightarrow J^t$ right continuous. Let further the family $(\mathcal{C}^t)_{t \in \mathbb{R}}$ of σ -algebras be *right continuous* in the sense that $\forall t \in \mathbb{R}, \mathcal{C}^t = \bigcap_{u>t} \mathcal{C}^u$. Then, there exists a function g on $\Omega \times \mathbb{R}$ with values in \mathbb{R}^+ such that g is a regular modification of f .

This theorem is proved in P.A. Meyer [1] in theorems *T4* and *T3* of Chap. VI, in pages 95 and 94. In that book, the supermartingales are assumed to take only real values. Hence to deduce the fundamental theorem from the theorems *T4* and *T3* mentioned above, we observe the following.

Since $\forall t, J^t < +\infty$, we have $\forall t, \forall \lambda w, f(w, t) < +\infty$. Define a function h on $\Omega \times \mathbb{R}$ with values in \mathbb{R}^+ as follows:

$$h(w, t) = \begin{cases} f(w, t) & \text{if } f(w, t) < +\infty. \\ 0 & \text{otherwise.} \end{cases}$$

Then, h is a modification of f with values in \mathbb{R}^+ . Hence by applying the theorems *T4* and *T3* of Chap. VI of P.A. Meyer [1], we get a regular modification for h and it is also a regular modification of f as well.

Remark 65. Let $(\mathcal{C}^t)_{t \in \mathbb{R}}$ be an increasing right continuous family of σ -algebras contained in $\hat{\mathcal{O}}_\lambda$. Let f be a right continuous supermartingale adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$, such that $\forall t \in \mathbb{R}, J^t = \int f(w, t) d\lambda(w) < +\infty$. Then f is regular.

For, by the fundamental theorem, there exists a regular modification g of f since $J^t < +\infty \forall t \in \mathbb{R}$ and $t \rightarrow J^t$ is right continuous by Fatou’s

lemma.

$$\begin{aligned} \forall t, \forall_{\lambda} w, g(w, t) &= f(w, t) \\ \forall_{\lambda} w, \forall t \in \mathbb{Q}, g(w, t) &= f(w, t). \end{aligned}$$

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Because $\forall_{\lambda} w$, g_w and f_w are right continuous, it follows that $\forall_{\lambda} w$, $\forall t \in \mathbb{R}$, $g(w, t) = f(w, t)$. Since $\forall_{\lambda} w$, g_w is regular, it follows that $\forall_{\lambda} w$, f_w is also regular.

The following procedure is standard and gives a method of finding out the regular modification of a supermartingale, whenever it exists.

Let $\left(\frac{k}{2^n}\right)_{k \in \mathbb{Z}, n \in \mathbb{N}}$ be the set of all dyadic rationals. Then, $\forall n \in \mathbb{N}$, $\mathbb{R} = \bigcup_{k \in \mathbb{Z}} \left(\frac{k}{2^n}, \frac{k+1}{2^n}\right]$. $\forall n \in \mathbb{N}$, define the functions τ_n from \mathbb{R} to \mathbb{R} as follows:

$$\text{If } t \in \mathbb{R}, \tau_n(t) = \frac{k+1}{2^n} \text{ if } t \in \left(\frac{k}{2^n}, \frac{k+1}{2^n}\right].$$

Then $\forall t \in \mathbb{R}$, $\tau_n(t) \downarrow t$ as $n \rightarrow \infty$ and if t is a dyadic rational, $\tau_n(t) = t$ for all n large enough.

Let f be a function on $\Omega \times \mathbb{R}$ with values in $\bar{\mathbb{R}}^+$. Define another function \bar{f} on $\Omega \times \mathbb{R}$ with values in $\bar{\mathbb{R}}^+$ as follows:

$$\bar{f}(w, t) = \begin{cases} \lim_{n \rightarrow \infty} f^{\tau_n(t)}(w), & \text{if the limit exists and is finite.} \\ 0, & \text{otherwise.} \end{cases}$$

It is obvious that if t is a dyadic rational and if $f(w, t) < +\infty$, then $\bar{f}(w, t) = f(w, t)$.

Proposition 66. *Let f be a function on $\Omega \times \mathbb{R}$ with values in $\bar{\mathbb{R}}^+$. Then*

- (i) *If the family $(\mathcal{C}^t)_{t \in \mathbb{R}}$ is right continuous, and if f is adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$, so is \bar{f} .*
- (ii) *If f is right continuous, and if $\forall_{\lambda} w$, $\forall t$, $f(w, t) < +\infty$, then $\forall_{\lambda} w$, $\forall t$, $\bar{f}(w, t) = f(w, t)$.*

Proof. (i) Since f is adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$ and $(\mathcal{C}^t)_{t \in \mathbb{R}}$ is right continuous, we can easily see that $\forall t$, the set $\{w : \lim_{n \rightarrow \infty} f^{\tau_n(t)}(w) \text{ exists and is finite}\}$ belongs to τ^t . From this, it follows that $\forall t, \bar{f}^t \in \mathcal{C}$ and this means that \bar{f} is adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$.

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(ii) Since f is right continuous, there exists a set $N_1 \in \mathcal{O}$ with $\lambda(N_1) = 0$ such that if $w \notin N_1$, f_w is a right continuous function on \mathbb{R} . Let $N_2 \in \mathcal{O}$, $\lambda(N_2) = 0$ be such that if $w \notin N_2$, $f(w, t) < +\infty$ for all $t \in \mathbb{R}$. Then if $w \notin N_1 \cup N_2$, for all t , $\lim_{n \rightarrow \infty} f^{\tau_n(t)}(w)$ exists and is equal to $f(w, t)$ and hence is finite. Thus, if $w \notin N_1 \cup N_2$, $\bar{f}(w, t) = f(w, t) \forall t \in \mathbb{R}$.

Hence, $\forall \lambda w, \forall t \in \mathbb{R}, \bar{f}(w, t) = f(w, t)$. □

We have the following obvious corollary.

Corollary 67. *Let $(\mathcal{C}^t)_{t \in \mathbb{R}}$ be a right continuous increasing family of sub σ -algebras of $\hat{\mathcal{O}}_\lambda$. Let f be a supermartingale adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$ and let g be a regular modification of f . Then, $\forall \lambda w, \forall t, g(w, t) = \bar{f}(w, mt)$. In particular, if f is a supermartingale adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$ with $J^t < +\infty$ and $t \rightarrow J^t$ right continuous, then $\forall \lambda w, \forall t, \lim_{n \rightarrow \infty} f(w, \tau_n(t))$ exists and is finite and $\forall t, \forall \lambda w, f(w, t) = \bar{f}(w, t)$. Moreover, \bar{f} is a regular modification of f .*

In the course of proofs of several theorems in this book, the following important theorem will be needed. We call it the “Upper envelope theorem” and whenever we refer to it, we will refer to it as the “Upper envelope theorem”. We state this without proof.

Let (X, \mathfrak{X}, μ) be a measure space. Let $(\mathfrak{X}^t)_{t \in \mathbb{R}}$ be an increasing right continuous family of σ -algebras contained in $\hat{\mathfrak{X}}_\mu$. Let $(f_n)_{n \in \mathbb{N}}$ be an increasing sequence of regular supermartingales in the sense that $\forall n \in \mathbb{N}$, f_n is a regular super martingale adapted to $(\mathfrak{X}^t)_{t \in \mathbb{R}}$ such that $\forall \mu x, \forall t, f_n(x, t) \leq f_{n+1}(x, t)$ for all $n \in \mathbb{N}$. Let $f(x, t) = \sup_n f_n(x, t)$. Then f is a right continuous supermartingale and $\forall \mu x$, limits from the left exist at all points.

For a proof see P.A. Meyer [1]. T16, p.99.

2 Measure valued Supermartingales

74 Let (Y, \mathcal{Y}) be a measurable space. Let ν be a measure valued function on $\Omega \times \mathbb{R}$ with values in $m^+(Y, \mathcal{Y})$, taking a point (w, t) of $\Omega \times \mathbb{R}$ to the measure ν_w^t on \mathcal{Y} . Let $\forall t \in \mathbb{R}$, ν^t be the measure valued function on Ω with values in $m^+(Y, \mathcal{Y})$ taking $w \in \Omega$ to ν_w^t and $\forall w \in \Omega$, let ν_w be the measure valued function on \mathbb{R} taking $t \in \mathbb{R}$ to the measure ν_w^t . If f is a function on Y , $f \geq 0$, $f \in \mathcal{Y}$, let $\nu(f)$ be the function on $\Omega \times \mathbb{R}$ with values in $\bar{\mathbb{R}}^+$ taking (w, t) to $\nu_w^t(f)$, $\forall w \in \Omega$, let $\nu_w(f)$ be the function on \mathbb{R} taking t to $\nu_w^t(f)$ and $\forall t \in \mathbb{R}$, let $\nu^t(f)$ be the function on Ω taking $w \in \Omega$ to $\nu_w^t(f)$.

Definition 68. ν is said to be a measure valued supermartingale (resp. measure valued martingale) adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$ if

$$(i) \quad \forall t, \nu^t \in \mathcal{C}^t$$

$$(ii) \quad \forall s, t \in \mathbb{R}, s \leq t \text{ and } \forall A \in \mathcal{C}^s \text{ and}$$

$$\int_A \nu_w^t d\lambda(w) \leq \int_A \nu_w^s d\lambda(w)$$

(resp.

$$(i) \quad \forall t, \nu^t \in \mathcal{C}^t$$

$$(ii) \quad \forall s, t \in \mathbb{R}, s \leq t \text{ and } \forall A \in \mathcal{C}^s,$$

$$\int_A \nu_w^t d\lambda(w) = \int_A \nu_w^s d\lambda(w))$$

in the sense that \forall function f on Y , $f \in \mathcal{Y}$, $f \geq 0$,

$$\int_A \nu_w^t(f) d\lambda(w) \geq \int_A \nu_w^s(f) d\lambda(w)$$

(resp. $\int_A \nu_w^t(f) d\lambda(w) = \int_A \nu_w^s(f) d\lambda(w)$.)

It is clear that ν is a measure valued supermartingale (resp. measure valued martingale) adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$ if and only if \forall function f on Y , $f \in \mathcal{Y}$, $f \geq 0$, $\nu(f)$ is an extended real valued supermartingale (resp. martingale) adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$.

Definition 69. ν is said to be a regular measure valued supermartingale adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$ if $\forall B \in \mathcal{Y}$, $\nu(\chi_B)$ is a regular supermartingale adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$. 75

Definition 70. If ν and μ are two measure valued supermartingales, adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$, with values in $m^+(Y, \mathcal{Y})$, μ is said to be a modification of ν if $\forall t, \forall \lambda w, \mu^t = \nu^t$.

Definition 71. If ν and μ are two measure valued supermartingales with values in $m^+(Y, \mathcal{Y})$ and adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$, μ is said to be a regular modification of ν if μ is regular and is a modification of ν .

Let ν be a measure valued supermartingale adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$, with values in $m^+(Y, \mathcal{Y})$. Let $\forall t \in \mathbb{R}, J^t = \int \nu_w^t d\lambda(w)$. Then, $\forall t, J^t$ is a positive measure on \mathcal{Y} . Since ν is a measure valued supermartingale, $\forall s, t \in \mathbb{R}, s \leq t$, we have $J^t \leq J^s$ in the sense that $\forall B \in \mathcal{Y}, J^t(B) \leq J^s(B)$. In this sense we say that $t \rightarrow J^t$ is decreasing. Let $\sup_{t \in \mathbb{R}} J^t$ be the set function defined on \mathcal{Y} as, $\forall B \in \mathcal{Y}, (\sup_{t \in \mathbb{R}} J^t)(B) = \sup_{t \in \mathbb{R}} J^t(B)$. With a similar definition for the set functions, $\sup_{t \in \mathbb{Z}} J^t, \sup_{\substack{t \in \mathbb{Z} \\ t \leq 0}} J^t$, we have $\sup_{t \in \mathbb{R}} J^t = \sup_{t \in \mathbb{Z}} J^t = \sup_{\substack{t \in \mathbb{Z} \\ t \leq 0}} J^t$ since $t \rightarrow J^t$ is decreasing. $\sup_{t \in \mathbb{Z}} J^t$ is a measure on \mathcal{Y} , since it is the supremum of an increasing sequence of measures on \mathcal{Y} . Let us denote this measure by J . Thus,

$$J = \sup_{t \in \mathbb{R}} J^t = \sup_{t \in \mathbb{Z}} J^t = \sup_{\substack{t \in \mathbb{Z} \\ t \leq 0}} J^t.$$

3 Properties of regular measure valued supermartingales

In this section, let ν be a regular measure valued supermartingale with values in $m^+(Y, \mathcal{Y})$, adapted to an increasing right continuous family $(\mathcal{C}^t)_{t \in \mathbb{R}}$ of σ -algebras contained in $\hat{\mathcal{O}}_\lambda$. Let $J^t = \int \nu_w^t d\lambda(w)$ and $J = \sup_{t \in \mathbb{R}} J^t$.

76 Proposition 72. *Let f be a function on Y , $f \geq 0$, $f \in \mathcal{Y}$ such that f is J -integrable. Then, $\nu(f)$ is a regular supermartingale with values in $\bar{\mathbb{R}}^+$, adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$ and $\forall \lambda w, \forall t$, f is ν_w^t -integrable.*

Proof. We know that $\forall B \in \mathcal{Y}$, $\nu(\chi_B)$ is a regular supermartingale, with values in $\bar{\mathbb{R}}^+$ adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$. Hence if s is a step function on Y , $s \geq 0$, $s \in \mathcal{Y}$, $\nu(s)$ is again a regular supermartingale with values in $\bar{\mathbb{R}}^+$ and adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$. Since f is J -integrable, $f \in \mathcal{Y}$ and $f \geq 0$, there exists an increasing sequence $(s_n)_{n \in \mathbb{N}}$ of step functions such that $\forall n \in \mathbb{N}$, $0 \leq s_n \leq f$, $s_n \in \mathcal{Y}$ and $s_n(y) \uparrow f(y)$ for all $y \in Y$.

Hence $\forall t \in \mathbb{R}$, $\forall w \in \Omega$, $\nu_w^t(s_n) \uparrow \nu_w^t(f)$. Hence $(\nu(s_n))_{n \in \mathbb{N}}$ is an increasing sequence of supermartingales with values in $\bar{\mathbb{R}}^+$, adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$, with limit as $\nu(f)$. Hence, by the 'Upper envelope theorem'. $\nu(f)$ is a right continuous supermartingale. $\forall t$, $\int \nu_w^t(f) d\lambda(w) = J^t(f) \leq J(f) < +\infty$. Hence by remark (4, § 1, 65), $\nu(f)$ is a regular supermartingale. Hence $\forall \lambda w, \forall t$, $\nu_w^t(f) < +\infty$ and this proves that $\forall \lambda w, \forall t$, f is ν_w^t -integrable. \square

Corollary 73. *Let f be a function on Y , with values in $\bar{\mathbb{R}}$, $f \in \mathcal{Y}$ and J -integrable. Then, $\forall \lambda w, \forall t$, f is ν_w^t -integrable and $\forall \lambda w$, the function $\nu_w(f)$ on \mathbb{R} taking t to $\nu_w^t(f)$ is regulated and right continuous.*

Proof. This follows immediately from the above proposition (4, § 3, 72) by writing f as $f^+ - f^-$ with the usual notation. \square

Proposition 74. *Let E be a Banach space over \mathbb{R} . Let g be a step function on Y with values in E , $g \in \mathcal{Y}$ and J -integrable. Then, $\forall \lambda w, \forall t$, g is ν_w^t -integrable and $\forall \lambda w$, $\nu_w(g)$, the function on \mathbb{R} taking t to $\nu_w^t(g)$, is a right continuous and regulated function on \mathbb{R} with values in E . (A*

Banach space valued function f on \mathbb{R} is said to be regulated if $\forall t \in \mathbb{R}$, $\lim_{\substack{s \rightarrow t \\ s > t}} f(s)$ and $\lim_{\substack{s \rightarrow t \\ s < t}} f(s)$ exist in the Banach space).

Proof. Let $g = \sum_{i=1}^n \chi_{A_i} x_i$ where $\forall i, i = 1, \dots, n, x_i \in E, A_i \in \mathcal{Y}$ and $A_i \cap A_j = \emptyset$ if $i \neq j$. Since g is J -integrable, $\forall i = 1, \dots, n, J(A_i) < +\infty$. Hence, by proposition (4, § 3, 72), $\forall i = 1, \dots, n, \nu(A_i)$ is a regular supermartingale and hence $\forall \lambda w, \forall t, i = 1, \dots, n, \nu_w^t(A_i) < +\infty$ and $\forall \lambda w, \forall i = 1, \dots, n, \nu_w(A_i)$ is a right continuous regulated function on \mathbb{R} . 77

Hence $\forall \lambda w, \forall t, g$ is ν_w^t -integrable and $\nu_w^t(g) = \sum_{i=1}^n \nu_w^t(A_i) x_i$ for these w and for all t for which g is ν_w^t -integrable.

Thus, $\forall \lambda w, \nu_w(g) = \sum \nu_w(A_i) x_i$ on \mathbb{R} . Hence $\forall \lambda w, \nu_w(g)$ is a right continuous regulated function on \mathbb{R} . \square

Proposition 75. *Let E be a Banach space over \mathbb{R} and let f be a function on Y with values in $E, f \in \mathcal{Y}$ and J -integrable. Then, $\forall \lambda w, \forall t, f$ is ν_w^t -integrable and $\forall \lambda w, \nu_w(f)$ is a right continuous regulated function on \mathbb{R} with values in E .*

Proof. Since f is J -integrable, and $\in \mathcal{Y}, \forall n, \exists$ a step function g_n on $Y, g_n \in \mathcal{Y}$ such that $\int |g_n - f| dJ \leq \frac{1}{2^n}$.

Let $h = \sum_{n=1}^{\infty} 2^n |g_n - f|$.

Then h is ≥ 0 and is J -integrable on Y .

$$\forall n \in \mathbb{N}, |f| \leq |g_n - f| + |g_n| \leq \frac{h}{2^n} + |g_n|$$

By proposition (4, § 3, 72), $\forall \lambda w, \forall t, h$ is ν_w^t -integrable and by proposition (4, § 3, 74) $\forall \lambda w, \forall n \in \mathbb{N}, g_n$ is ν_w^t -integrable. Hence, $\forall \lambda w, \forall t, f$ is ν_w^t -integrable.

$$|g_n - f| \leq \frac{1}{2^n} h.$$

Hence $\forall \lambda w, \forall t, |\nu_w^t(g_n) - \nu_w^t(f)| \leq \frac{1}{2^n} \nu_w^t(h)$. Since h is ≥ 0 and J -integrable, by proposition (4, § 3, 72), $\forall \lambda w, \nu_w(h)$ is a right continuous

regulated function on \mathbb{R} . Hence $\forall_{\lambda} w, v_w(h)$ is locally bounded on \mathbb{R} . 78
 Hence $\forall_{\lambda} w, v_w^t(g_n)$ converges to $v_w^t(f)$ in E , as $n \rightarrow \infty$, locally uniformly in the variable t .

By the previous proposition (4, § 3, 74), $\forall_{\lambda} w, \forall n \in \mathbb{N}, v_w(g_n)$ is right continuous and regulated. Hence, since E is complete, and since $\forall_{\lambda} w$, the convergence of $v_w^t(g_n)$ to $v_w^t(f)$ is locally uniform in t , it follows that $v_w(f)$ is also right continuous and regulated. \square

Chapter 5

Existence and Uniqueness of regular modifications of measure valued supermartingales

1 Uniqueness theorem

Throughout this section, let $(\Omega, \mathcal{O}, \lambda)$ be a measure space. Let (Y, \mathcal{Y}) be a measurable space. Let ν be a measure valued supermartingale on $\Omega \times \mathbb{R}$ with values in $m^+(Y, \mathcal{Y})$ adapted to an increasing right continuous family $(\mathcal{C}^t)_{t \in \mathbb{R}}$ of sub σ -algebras of $\hat{\mathcal{O}}_\lambda$. Let ν take a point (w, t) of $\Omega \times \mathbb{R}$ to the measure ν_w^t on \mathcal{Y} . Let $\forall t, J^t = \int \nu_w^t d\lambda(w)$ and let $J = \sup_{t \in \mathbb{R}} J^t$. 79

Theorem 76 (Uniqueness). *Let J be a finite measure on \mathcal{Y} and let \mathcal{Y} have the J -countability property. If χ and Ψ are two regular modifications of ν , then $\forall \lambda w, \forall t, \chi_w^t = \Psi_w^t$.*

Proof. Since χ and Ψ are regular, $\chi(1)$ and $\Psi(1)$ are regular supermartingales. Hence $\forall \lambda w, \forall t, \chi_w^t(1)$ and $\Psi_w^t(1)$ are finite. Hence $\forall \lambda w, \forall t, \chi_w^t$ and Ψ_w^t are finite measures on \mathcal{Y} . Thus, there exists a set $N_1 \in \mathcal{O}$ with $\lambda(N_1) = 0$ such that if $w \notin N_1, \chi_w^t$ and Ψ_w^t are finite measures on \mathcal{Y} for

all $t \in \mathbb{R}$.

Since \mathcal{Y} has the J -countability property, there exists a set $\mathbb{N} \in \mathcal{Y}$ such that the σ -algebra $\mathcal{Y}' = \mathcal{Y} \cap \mathbb{C}N$ on $Y' = Y \cap \mathbb{C}N$ is countably generated. Let \mathcal{B} be a countable class generating \mathcal{Y}' . Without loss of generality, we can assume that \mathcal{B} is a π -system, containing Y' .

Let $B \in \mathcal{B}$. The supermartingales $\chi(B)$ and $\Psi(B)$ are both regular modifications of the supermartingale $\nu(B)$. Hence, by remark (4, § 1, 64),

$$\forall \lambda w, \forall t, \chi_w^t(B) = \Psi_w^t(B).$$

Thus, $\forall B \in \mathcal{B}, \forall \lambda w, \forall t, \chi_w^t(B) = \Psi_w^t(B)$.

80 Since \mathcal{B} is countable,

$$\forall \lambda w, \forall B \in \mathcal{B}, \forall t, \chi_w^t(B) = \Psi_w^t(B).$$

Hence there exists a set $N_2 \in \mathcal{O}$, with $\lambda(N_2) = 0$ such that if $w \notin N_2$, $\forall t \in \mathbb{R}, \forall B \in \mathcal{B}, \chi_w^t(B) = \Psi_w^t(B)$. Let $w \in \Omega$. Consider the class \mathcal{C}_w of all sets $C \in \mathcal{Y}'$ such that $\forall t, \chi_w^t(C) = \Psi_w^t(C)$. If $w \notin N_1 \cup N_2$, the class \mathcal{C}_w is a d -system containing the π -system \mathcal{B} . Hence, by the Monotone class theorem, for $w \notin N_1 \cup N_2$, \mathcal{C}_w contains the σ -algebra generated by \mathcal{B} which is \mathcal{Y}' . Thus, if $w \notin N_1 \cup N_2$, $\forall A \in \mathcal{Y}', \forall t, \chi_w^t(A) = \Psi_w^t(A)$.

Now, since $J(N) = 0, \forall t, J^t(N) = 0$.

$$J^t(N) = \int \chi_w^t(N) d\lambda(w).$$

Hence $\forall t, \forall \lambda w, \chi_w^t(N) = 0$. Therefore, $\forall \lambda w, \forall t \in \mathbb{Q}, \chi_w^t(N) = 0$. Since $\forall \lambda w, t \rightarrow \chi_w^t(N)$ is right continuous, it follows that,

$$\forall \lambda w, \forall t, \chi_w^t(N) = 0.$$

Similarly, $\forall \lambda w, \forall t, \Psi_w^t(N) = 0$.

Hence, there exists a set $N_3 \in \mathcal{O}$ with $\lambda(N_3) = 0$ such that if $w \notin N_3$,

$$\forall t, \chi_w^t(N) = \Psi_w^t(N) = 0.$$

Let $w \notin N_1 \cup N_2 \cup N_3$. Let $A \in \mathcal{Y}$.

$$A = A \cap Y' \cup A \cap N.$$

$$\begin{aligned}
\text{If } t \in \mathbb{R}, \chi_w^t(A) &= \chi_w^t(A \cap Y') \\
&= \Psi_w^t(A \cap Y') \\
&= \Psi_w^t(A).
\end{aligned}$$

Hence, if $w \notin N_1 \cup N_2 \cup N_3$, $\forall t \in \mathbb{R}$, $\forall A \in \mathcal{Y}$, $\chi_w^t(A) = \Psi_w^t(A)$.
Therefore,

$$\forall \lambda w, \forall t, \chi_w^t = \Psi_w^t \text{ on } \mathcal{Y}.$$

□

2 Existence theorem

Throughout this section, let us assume that $(\Omega, \mathcal{O}, \lambda)$ is a measure space, **81**
 $(\mathcal{C}^t)_{t \in \mathbb{R}}$ is an increasing right continuous family of σ -algebras contained
in $\hat{\mathcal{O}}_\lambda$, Y is a topological space, \mathcal{Y} is its Borel σ -algebra, ν is a measure
valued supermartingale on $\Omega \times \mathbb{R}$ with values in $m^+(Y, \mathcal{Y})$, adapted to
 $(\mathcal{C}^t)_{t \in \mathbb{R}}$, $J^t = \int \nu_w^t d\lambda(w)$ and $J = \sup_{t \in \mathbb{R}} J^t$.

Definition 77. We say $t \rightarrow J^t$ is right continuous if \forall function f on Y ,
 $f \geq 0$, $f \in \mathcal{Y}$, $t \rightarrow J^t(f)$ is right continuous.

Theorem 78 (Existence). Let J be a finite measure on \mathcal{Y} and let $t \rightarrow J^t$
be right continuous. Let Y have the J -compactness metrizability property.
Then, there exists a regular modification of ν .

Proof. Since J is a finite measure, $\forall t$, J^t is also a finite measure. Hence
 $\forall t$, $\forall \lambda w$, ν_w^t is a finite measure. Define the measure valued function ν'
on $\Omega \times \mathbb{R}$ as

$$\nu_w^t = \begin{cases} \nu_w^t, & \text{if } \nu_w^t \text{ is a finite measure} \\ 0, & \text{otherwise.} \end{cases}$$

Then, ν' is a modification of ν and for all $w \in \Omega$, for all $t \in \mathbb{R}$, ν_w^t is
a finite measure on \mathcal{Y} . ν' , being a modification of ν , has the same $J^t \forall t$
and the same J . Hence, if we can prove the theorem for ν' under the
assumptions mentioned in the statement of the theorem, the theorem for

ν is obvious. Hence, without loss of generality, we shall assume that $\forall t, \forall w, \nu_w^t$ is a finite measure on \mathcal{Y} .

Let us divide the proof in two cases Case I and Case II. In Case I, we shall prove the theorem assuming Y to be a compact metrizable space and in case II, we shall deal with the general case and deduce the result from Case I.

82 In Case I the proof is carried out in four steps, Step I, Step II, Step III and Step IV.

In Step I, we define a measure valued function $\tilde{\nu}$ on $\Omega \times \mathbb{R}$ with values in ${}^+(Y, \mathcal{Y})$ such that $\forall t \in \mathbb{R}$ and $\forall w \in \Omega$, $\tilde{\nu}_w^t$ is a Radon measure on Y .

In Step II, we prove that $\forall t, \tilde{\nu}^t \in \mathcal{C}^t$.

In Step III, we prove that if $\mathcal{C}(Y)$ is the space of all real valued continuous functions on Y , $\forall \varphi \in \mathcal{C}(Y)$, $\forall \lambda w$, the function $\tilde{\nu}_w(\varphi)$ on \mathbb{R} taking $t \in \mathbb{R}$ to $\tilde{\nu}_w^t(\varphi)$, is regular and $\tilde{\nu}$ is a modification of ν .

In Step IV, we prove that $\tilde{\nu}$ is a regular measure valued supermartingale.

Case I. Y , a compact metrizable space.

Since J, J^t are finite measures on \mathcal{Y} , they are Radon measures on Y , since Y is a compact metrizable space. Let $\mathcal{C}(Y)$ be the vector space of all real valued continuous functions on Y and $\mathcal{C}_+(Y)$, the cone of all the positive real valued continuous functions on Y .

Step I. Let $\varphi \in \mathcal{C}_+(Y)$. Consider the real valued supermartingale $\nu(\varphi)$. $\forall t, \int \nu_w^t(\varphi) d\lambda(w) = J^t(\varphi)$ is finite and by hypothesis, $t \rightarrow J^t(\varphi)$ is right continuous. Hence, by the fundamental theorem, a regular modification for $\nu(\varphi)$ exists. Hence, by Corollary (4, § 1, 67), $\forall \lambda w, \forall t, \lim_{n \rightarrow \infty} \nu_w^{\tau_n(t)}(\varphi)$ exists and is finite, and if $\overline{\nu(\varphi)}(w, t) = \lim_{n \rightarrow \infty} \nu_w^{\tau_n(t)}(\varphi)$ if this limit exists and is finite, and = 0 otherwise, then, $\overline{\nu(\varphi)}$ is a regular modification of $\nu(\varphi)$.

Hence, if $\varphi \in \mathcal{C}(Y)$, $\forall \lambda w, \forall t, \lim_{n \rightarrow \infty} \nu_w^{\tau_n(t)}(\varphi)$ exists and is finite.

Thus, for $\varphi \in \mathcal{C}(Y)$, if

$$\Omega_{t,\varphi}^\circ = \{w \in \Omega \mid \lim_{n \rightarrow \infty} \nu_w^{\tau_n(t)}(\varphi) \text{ exists and is finite} \}.$$

and if $\Omega_\varphi^\circ = \bigcap_{t \in \mathbb{R}} \Omega_{t,\varphi}^\circ$, then λ is carried by Ω_φ° and hence a priori by $\Omega_{t,\varphi}^\circ$,

$\forall t \in \mathbb{R}$. Also, $\forall \varphi \in \mathcal{C}(Y)$, $\forall t \in \mathbb{R}$, $\Omega_{t,\varphi}^\circ \in \mathcal{C}^t$.

Let D be a countable dense subset of $\mathcal{C}(Y)$, containing 1. Let $\Omega^\circ = \bigcap_{\varphi \in D} \Omega_\varphi^\circ$ and $\Omega_t^\circ = \bigcap_{\varphi \in D} \Omega_{t,\varphi}^\circ$. Note that $\Omega^\circ = \bigcap_{t \in \mathbb{R}} \Omega_t^\circ$. 83

Since D is countable, λ is carried by Ω° , $\Omega_t^\circ \in \mathcal{C}^t \quad \forall t \in \mathbb{R}$ and λ is carried by Ω_t° for every $t \in \mathbb{R}$.

Let $w \in \Omega_t^\circ$. Since $1 \in D$, $\lim_{n \rightarrow \infty} \nu_w^{\tau_n(t)}(1)$ exists and is finite. Hence $\forall t$, $\sup_n \nu_w^{\tau_n(t)}(1)$ is finite. (1)

Also if $w \in \Omega_t^\circ$, $\forall \varphi \in D$, $\lim_{n \rightarrow \infty} \nu_w^{\tau_n(t)}(\varphi)$ exists and is finite. (2)

From (1) and (2), we can easily deduce that if $w \in \Omega_t^\circ$, then $\forall \varphi \in \mathcal{C}(Y)$, $\lim_{n \rightarrow \infty} \nu_w^{\tau_n(t)}(\varphi)$ exists and is finite. Thus,

$$\Omega_t^\circ = \bigcap_{\varphi \in \mathcal{C}(Y)} \Omega_{t,\varphi}^\circ.$$

In the same way,

$$\Omega^\circ = \bigcap_{\varphi \in \mathcal{C}(Y')} \Omega_\varphi^\circ.$$

Thus, if $w \in \Omega_t^\circ$, the vague limit of $\nu_w^{\tau_n(t)}$ exists. Conversely if for a $w \in \Omega$ and for a $t \in \mathbb{R}$, the vague limit of $\nu_w^{\tau_n(t)}$ exists, then it is easy to see that $w \in \Omega_t^\circ$. Thus,

$$\Omega_t^\circ = \{w \in \Omega \mid \text{vague limit of } \nu_w^{\tau_n(t)} \text{ exists} \}.$$

Define the measure valued function $\tilde{\nu}$ on $\Omega \times \mathbb{R}$ with values in $\mathfrak{m}^+(Y, \mathcal{Y})$ as follows.

$$\tilde{\nu}_w^t = \begin{cases} \text{vague limit of } \nu_w^{\tau_n(t)}, & \text{if } w \in \Omega_t^\circ \\ 0, & \text{otherwise.} \end{cases}$$

$\tilde{\nu}_w^t$ is thus a Radon measure on Y for all $w \in \Omega$ and for all $t \in \mathbb{R}$.

Step II. Let show that $\forall t$, $\tilde{\nu}^t \in \mathcal{C}^t$.

For this, we have to show that $\forall B \in \mathcal{Y}$, the function $\tilde{\nu}^t(B) \in \mathcal{C}^t$.

Let us first show that if $\varphi \in \mathcal{C}_+(Y)$, then $\tilde{\nu}^t(\varphi) \in \mathcal{C}^t$. 84

$$\tilde{v}_w^t(\varphi) = \begin{cases} \lim_{n \rightarrow \infty} v_w^{\tau_n(t)}(\varphi), & \text{if } w \in \Omega_t^\circ \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $\tilde{v}_w^t(\varphi) = \chi_{\Omega_t^\circ} \limsup_{n \rightarrow \infty} v_w^{\tau_n(t)}(\varphi)$. Since $\Omega_t^\circ \in \mathcal{C}^t$ and since $w \rightarrow \limsup_{n \rightarrow \infty} v_w^{\tau_n(t)}(\varphi)$ belongs to \mathcal{C}^t , it follows that $\tilde{v}^t(\varphi) \in \mathcal{C}^t$.

Let U be an open subset of Y , $U \neq \emptyset$. Then, since Y is metrizable, there exists an increasing; sequence $(\varphi_n)_{n \in \mathbb{N}}$ of continuous functions on Y such that $0 \leq \varphi_n \leq \chi_n \forall n \in \mathbb{N}$ and $\varphi_n(y) \uparrow \chi_U(y)$ for all $y \in Y$.

Hence, $\forall t \in \mathbb{R}, \forall w \in \Omega, \tilde{v}_w^t(\chi_U) = \lim_{n \rightarrow \infty} \tilde{v}_w^t(\varphi_n)$. Since $\forall n \in \mathbb{N}, \forall t \in \mathbb{R}, \tilde{v}^t(\varphi_n) \in \mathcal{C}^t$, it follows that $\forall t \in \mathbb{R}, \tilde{v}^t(\chi_U)$ also belongs to \mathcal{C}^t .

Now, the standard application of the Monotone class theorem gives that $\forall B \in \mathcal{Y}, \forall t \in \mathbb{R}, \tilde{v}^t(\chi_B) \in \mathcal{C}^t$. Hence $\forall t \in \mathbb{R}, \tilde{v}^t \in \mathcal{C}^t$.

Step III. Let us shown that $\forall \varphi \in \mathcal{C}(Y), \forall \lambda w, \tilde{v}_w(\varphi)$ is regular and that \tilde{v} is a modification of v .

Let $w \in \Omega^\circ$. Then $\forall \varphi \in \mathcal{C}(Y), \forall t, \lim_{n \rightarrow \infty} v_w^{y_n(t)}(\varphi)$ exists and is finite and this limit is equal to $\tilde{v}_w^t(\varphi)$. Therefore, if $\varphi \in \mathcal{C}_+(Y)$ and $w \in \Omega^\circ$,

$$\tilde{v}_w^t(\varphi) = \overline{v(\varphi)}(w, t)$$

for all t . Since λ is carried by Ω° , we therefore have

$$\forall \lambda w, \forall t, \forall \varphi \in \mathcal{C}_+(Y), \tilde{v}_w^t(\varphi) = \overline{v(\varphi)}(w, t). \quad (1)$$

Now, $\overline{v(\varphi)}$ is a regular modification of $v(\varphi)$ by Corollary (4, § 1, 67). Hence, $\forall \lambda w, \overline{v(\varphi)}_w$ is regular. Therefore, since from (1).

$$\forall \lambda w, \forall t, \forall \varphi \in \mathcal{C}_+(Y), \tilde{v}_w^t(\varphi) = \overline{v(\varphi)}(w, t),$$

85 it follows that, $\forall \varphi \in \mathcal{C}_+(Y), \forall \lambda w, \tilde{v}_w(\varphi)$ is regular. Hence $\forall \varphi \in \mathcal{C}(Y), \forall \lambda w, \tilde{v}_w(\varphi)$ is also regular.

If $\varphi \in \mathcal{C}_+(Y)$, since $\overline{v(\varphi)}$ is a modification of $v(\varphi)$, we have

$$\forall \varphi \in \mathcal{C}_+(Y), \quad \forall \lambda w, \overline{v(\varphi)}(w, t) = v_w^t(\varphi).$$

Hence from this and from (1), we deduce that

$$\forall \varphi \in \mathcal{C}_+(Y), \forall t, \quad \forall \lambda w, \tilde{v}_w^t(\varphi) = v_w^t(\varphi).$$

Hence $\forall \varphi \in \mathcal{C}(Y), \forall t, \forall \lambda w, \tilde{v}_w^t(\varphi) = v_w^t(\varphi)$. Therefore, since D is countable,

$$\forall t, \quad \forall \lambda w, \tilde{v}_w^t(\varphi) = v_w^t(\varphi)$$

for all $\varphi \in D$.

Since D is dense and the measures \tilde{v}_w^t and v_w^t are finite measures for all $w \in \Omega$ and for all $t \in \mathbb{R}$, it follows that

$$\forall t, \quad \forall \lambda w, \quad \forall \varphi \in \mathcal{C}(Y), \tilde{v}_w^t(\varphi) = v_w^t(\varphi)$$

and hence, $\forall t, \forall \lambda w, \tilde{v}_w^t = v_w^t$ as v_w^t and \tilde{v}_w^t are Radon measures for all $w \in \Omega$ and for all $t \in \mathbb{R}$.

Hence $\tilde{\nu}$ is a modification of ν .

Step IV.

We shall show that $\tilde{\nu}$ is regular. For this we have to show that $\forall B \in \mathcal{B}, \forall \lambda w, \tilde{\nu}_w(\chi_B)$ is regular.

From Step III, if $\varphi \in \mathcal{C}_+(Y), \forall \lambda w, \tilde{\nu}_w(\varphi)$ is regular. In particular, $\forall \lambda w, \tilde{\nu}_w(1)$ is regular. Hence, $\forall \lambda w, \tilde{\nu}_w(1)$ is a locally bounded function on \mathbb{R} i.e. \exists a set $N \in \mathcal{O}$ with $\lambda(N) = 0$ such that if $w \notin N, \tilde{\nu}_w(1)$ is a locally bounded function on \mathbb{R} . Hence if $B \in \mathcal{B}$ and $w \notin N, \tilde{\nu}_w(\chi_B)$ is also a locally bounded function on \mathbb{R} .

Let U be an open set of $Y, U \neq \emptyset$. Then since Y is metrizable, there exists an increasing sequence $(\varphi_n)_{n \in \mathbb{N}}$ of continuous functions on Y such that $\forall n \in \mathbb{N}, 0 \leq \varphi_n \leq \chi_U$ and

$$\varphi_n(y) \uparrow \chi_U(y) \text{ for all } \underline{y \in Y}.$$

Hence $\forall w \in \Omega, \forall t \in \mathbb{R}, \tilde{\nu}_w^t(\varphi_n) \uparrow \tilde{\nu}_w^t(\chi_U)$. Since $\forall n, \tilde{\nu}(\varphi_n)$ is a regular supermartingale and since $\tilde{\nu}(\varphi_n)$ is an increasing sequence of functions on $\Omega \times \mathbb{R}$, it follows from the 'Upper envelope theorem' that $\forall \lambda w, \tilde{\nu}_w(\chi_U)$ is right continuous and has limits from the left at all points $t \in \mathbb{R}$. The finiteness of these limits from the left at all $t \in \mathbb{R}$, follows from the fact that $\forall \lambda w, \tilde{\nu}_w(\chi_U)$ is locally bounded. Hence, $\forall \lambda w, \tilde{\nu}_w(\chi_U)$ is regular.

Let $\mathcal{C} = \{C \in \mathcal{B} \mid \forall \lambda w, \tilde{\nu}_w(\chi_C) \text{ is regular}\}$. Then, the class \mathcal{C} is a d -system again by the 'Upper envelope theorem' and by the local boundedness of $\tilde{\nu}_w(\chi_B) \quad \forall B \in \mathcal{B}$, for almost all $w \in \Omega$. This class \mathcal{C} contains the class \mathcal{U} of all open sets which is a π -system generating \mathcal{B} .

Hence, \mathcal{C} contains the σ -algebra \mathcal{Y} and hence is equal to \mathcal{Y} . Thus, $\forall B \in \mathcal{Y}$, $\forall \lambda w$, $\tilde{\nu}_w(\chi_B)$ is regular.

Since $\tilde{\nu}$ is a modification of ν , $\tilde{\nu}(\chi_B)$ is a supermartingale $\forall B \in \mathcal{Y}$. Moreover, since $\forall B \in \mathcal{Y}$, $\forall \lambda w$, $\tilde{\nu}_w(\chi_B)$ is regular, $\tilde{\nu}(\chi_B)$ is a regular supermartingale $\forall B \in \mathcal{Y}$. Hence, $\tilde{\nu}$ is a regular supermartingale and is a regular modification of ν .

Thus, the proof in case I is complete. Note that in this case, because of the uniqueness theorem, the regular modification of ν is unique.

Case II. Y , a general topological space having the J -compactiy metrizable property.

Since Y has the J -compactiy metrizable property, there exists a sequence $(X_n)_{n \in \mathbb{N}}$ of compact metrizable sets and a set $N \in \mathcal{Y}$ such that $J(N) = 0$ and $Y = \bigcup_{n \in \mathbb{N}} X_n \cup N$.

87 Let $\forall n \in \mathbb{N}$, $Y_n = \bigcup_{i=1}^n X_i$. Then $\forall n$, Y_n is again a compact metrizable set. This is because $\forall n$, Y_n is both compact and Suslin each X_i is so. Moreover, the sequence $(Y_n)_{n \in \mathbb{N}}$ is increasing and $Y = \bigcup_{n \in \mathbb{N}} Y_n \cup N$.

Let $\forall n \in \mathbb{N}$, \mathcal{Y}_n be the Borel σ -algebra of Y_n . Let $\forall n \in \mathbb{N}$, ν^n be the measure valued function on $\Omega \times \mathbb{R}$ with values in $m^+(Y_n, \mathcal{Y}_n)$ associating to each $(w, t) \in \Omega \times \mathbb{R}$, the measure $\nu_w^{n,t}$ on \mathcal{Y}_n , which is the restriction of the measure ν_w^t to Y_n . Then, it is easily seen that $\forall n$, ν^n is a measure valued supermartingale on $\Omega \times \mathbb{R}$ with values in $m^+(Y_n, \mathcal{Y}_n)$. Let $J_n^t = \int \nu_w^{n,t} d\lambda(w)$ and $J_n = \sup_{t \in \mathbb{R}} J_n^t$. Then, $\forall n \in \mathbb{N}$, J_n^t and J_n are respectively the restriction of J^t and J to Y_n . Hence $\forall n$, $\forall t$, J_n^t is finite, J_n is finite and $t \rightarrow J_n^t$ is right continuous. Hence the hypothesis of the theorem is verified for $\nu^n \forall n \in \mathbb{N}$. Hence by Case I, $\forall n \in \mathbb{N}$, \exists a *unique* measure valued supermartingale $\tilde{\nu}^n$ with values in $m^+(Y_n, \mathcal{Y}_n)$ taking (w, t) to $\tilde{\nu}_w^{n,t}$, which is a regular modification of ν^n .

Let $\forall n \in \mathbb{N}$, $\forall t$, $\tilde{\nu}_w^{n+1,t} | Y_n$ denote the restriction of the measure $\tilde{\nu}_w^{n+1,t}$ to Y_n and let $\tilde{\nu}_{Y_n}^{n+1}$ denote the measure valued function on $\Omega \times \mathbb{R}$ with values in $m^+(Y_n, \mathcal{Y}_n)$ associating (w, t) to $\tilde{\nu}_w^{n+1,t} | Y_n$. Then it is easily seen that $\forall n$, $\tilde{\nu}_{Y_n}^{n+1}$ is also a regular modification of ν^n . Hence, because of uniqueness,

$$\forall \lambda w, \forall t, \tilde{\nu}_w^{n,t} = \tilde{\nu}_w^{n+1,t} | Y_n \quad (1)$$

$\forall n \in \mathbb{N}$, let us consider the measures $\tilde{\nu}_w^{n,t}$ as measures on \mathcal{Y} by defining it to be zero for any set $B \in \mathcal{Y}$ for which $B \cap Y_n = \emptyset$.

Then from (1), it follows that $\forall \lambda w$, $\tilde{\nu}_w^{n,t}$ is an increasing sequence of measures on \mathcal{Y} for all $t \in \mathbb{R}$. i.e. there exists a set $N_1 \in \mathcal{O}$ with $\lambda(N_1) = 0$ such that if $w \notin N_1$, for all $t \in \mathbb{R}$, $(\tilde{\nu}_w^{n,t})_{n \in \mathbb{N}}$ is an increasing sequence of measures on \mathcal{Y} .

Now, $\forall n \in \mathbb{N}$, define the measure valued function μ^n on $\Omega \times \mathbb{R}$, with values in $m^+(Y, \mathcal{Y})$ taking (w, t) to $\mu_w^{n,t}$ as follows.

$$\mu_w^{n,t} = \begin{cases} \tilde{\nu}_w^{n,t}, & \text{if } w \notin N_1 \\ 0, & \text{if } w \in N_1 \end{cases}$$

Then $\forall w \in \Omega$, $\forall t \in \mathbb{R}$, $(\mu_w^{n,t})_{n \in \mathbb{N}}$ is an increasing sequence of measures of \mathcal{Y} . $\forall n \in \mathbb{N}$, μ^n is a regular measure valued supermartingale on $\Omega \times \mathbb{R}$ with values in $m^+(Y, \mathcal{Y})$. 88

Define $\forall w \in \Omega$, and $\forall t \in \mathbb{R}$, the measures μ_w^t as $\sup_n \mu_w^{n,t}$. Then

$$\forall B \in \mathcal{Y}, \mu_w^t(B) = \sup_n \mu_w^{n,t}(B).$$

Let μ be the measure valued function on $\Omega \times \mathbb{R}$ taking (w, t) to μ_w^t .

Since $\forall B \in \mathcal{Y}$, $\forall n \in \mathbb{N}$, $\mu^n(\chi_B)$ is a regular supermartingale, it follows from the 'Upper envelope theorem' that $\forall \lambda w$, $\mu_w(\chi_B)$ is right continuous and limits from the left exist at all $t \in \mathbb{R}$. Hence $\mu(\chi_B)$ is a right continuous supermartingale and $\forall t \in \mathbb{R}$,

$$\begin{aligned} \int \mu_w^t(\chi_B) d\lambda(w) &= \lim_{n \rightarrow \infty} \int \mu_w^{n,t}(\chi_B) d\lambda(w) \\ &= \lim_{n \rightarrow \infty} \int \tilde{\nu}_w^{n,t}(\chi_B) d\lambda(w) \\ &= \lim_{n \rightarrow \infty} \int \nu_w^{n,t}(\chi_B) d\lambda(w) \\ &= \lim_{n \rightarrow \infty} \int \nu_w^t(B \cap Y_n) d\lambda(w) \\ &= \lim_{n \rightarrow \infty} J^t(B \cap Y_n) \\ &= J^t(B) < +\infty. \end{aligned}$$

Hence by remark (4, § 1, 65), $\mu(\chi_B)$ is a regular supermartingale. Therefore μ is a regular measure valued supermartingale.

We have to show that μ is a modification of ν . i.e. we have to show that $\forall t, \forall \lambda w, \mu_w^t = \nu_w^t$.

89 Let $t \in \mathbb{R}$ be fixed. Since $J(N) = 0, J^t(N) = 0$. Hence $\forall \lambda w, \nu_w^t(N) = 0$ i.e. $\exists N_t^1 \in \mathcal{O}$ such that $\lambda(N_t^1) = 0$ and if $w \notin N_t^1$, then $\nu_w^t(N) = 0$. $\forall n \in \mathbb{N}, \tilde{\nu}^n$ is a modification of ν^n . Hence,

$$\forall n \in \mathbb{N}, \quad \forall \lambda w, \tilde{\nu}_w^{n,t} = \nu_w^{n,t}.$$

Therefore, $\forall \lambda w, \forall n \in \mathbb{N}, \tilde{\nu}_w^{n,t} = \nu_w^{n,t}$. Hence there exists a set $N_t^2 \in \mathcal{O}$ with $\lambda(N_t^2) = 0$ such that if $w \notin N_t^2$,

$$\tilde{\nu}_w^{n,t} = \nu_w^{n,t}$$

for all $n \in \mathbb{N}$,

Let $M = N_t^1 \cup N_t^2 \cup N_1$. Then $M \in \mathcal{O}$ and $\lambda(M) = 0$. If $w \notin M$, and if $B \in \mathcal{Y}$,

$$\begin{aligned} \nu_w^t(B) &= \lim_{n \rightarrow \infty} \nu_w^t(Y_n \cap B) \\ &= \lim_{n \rightarrow \infty} \nu_w^{n,t}(B) \\ &= \lim_{n \rightarrow \infty} \tilde{\nu}_w^{n,t}(B) \\ &= \lim_{n \rightarrow \infty} \mu_w^{n,t}(B) \\ &= \mu_w^t(B). \end{aligned}$$

Hence, if $w \notin M, \nu_w^t = \mu_w^t$. Therefore

$$\forall t, \quad \forall \lambda w, \mu_w^t = \nu_w^t.$$

This proves that μ is a modification of ν . □

3 The measures ν_w^t and the left limits

90 Throughout this section, let $(\Omega, \mathcal{O}, \lambda)$ be a measure space. Let $(\mathcal{C}^t)_{t \in \mathbb{R}}$ be an increasing right continuous family of σ -algebras contained in $\hat{\mathcal{O}}_\lambda$.

Let Y be a compact metrizable space and \mathcal{Y} be its Borel σ -algebra. Let ν be a regular measure valued supermartingale on $\Omega \times \mathbb{R}$ with values in $m^+(Y, \mathcal{Y})$, adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$. Let $\forall t \in \mathbb{R}$, $J^t = \int \nu_w^t d\lambda(w)$ and $J = \sup_{t \in \mathbb{R}} J^t$. Let us assume that J is a finite measure on \mathcal{Y} .

Since ν is a regular measure valued supermartingale, $\forall \lambda w$, $\nu_w(1)$ is a right continuous, regulated function on \mathbb{R} . Hence $\forall \lambda w$, $\nu_w(1)$ is locally bounded on \mathbb{R} . i.e. \exists a set $N_1 \in \mathcal{O}$ such that $\lambda(N_1) = 0$ and if $w \notin N_1$, $\nu_w(1)$ is locally bounded on \mathbb{R} .

$\forall \varphi \in \mathcal{C}(Y)$, the space of all real valued continuous functions on Y , $\forall \lambda w$, $\nu_w(\varphi)$ is a right continuous regulated function on \mathbb{R} . Hence, $\forall \varphi \in \mathcal{C}(Y)$, $\forall \lambda w$, $t \rightarrow \nu_w^t(\varphi)$ is right continuous and has finite left limits at all points of \mathbb{R} .

Let D be a countable dense subset of $\mathcal{C}(Y)$. We have,

$\forall \lambda w$, $\forall \varphi \in D$, $t \rightarrow \nu_w^t(\varphi)$ is right continuous and has finite left limits at all points of \mathbb{R} . i.e. there exists a set $N_2 \in \mathcal{O}$ such that if $w \notin N_2$, $\forall \varphi \in D$, $t \rightarrow \nu_w^t(\varphi)$ is right continuous and has finite left limits at all points of \mathbb{R} .

Let $w \notin N_1 \cup N_2$. Then it is easy to see that $\forall \varphi \in \mathcal{C}(Y)$, $t \rightarrow \nu_w^t(\varphi)$ is right continuous and has finite limits at all points of \mathbb{R} .

Hence, define $\forall w \in \Omega$, $\forall t \in \mathbb{R}$, the linear mappings ν_w^{t-} on $\mathcal{C}(Y)$, with values on \mathbb{R} as follows, $\forall \varphi \in \mathcal{C}(Y)$,

$$\nu_w^{t-}(\varphi) = \begin{cases} \lim_{\substack{s \rightarrow t \\ s < t}} \nu_w^s(\varphi), & \text{if } w \notin N_1 \cup N_2 \\ 0, & \text{if } w \in N_1 \cup N_2. \end{cases}$$

Then, $\forall w \in \Omega$, $\forall t \in \mathbb{R}$, ν_w^{t-} is a positive linear functional on $\mathcal{C}(Y)$ and thus defines a Radon measure on Y . And, by definition,

$$\forall \lambda w, \forall t, \nu_w^{t-}(\varphi) = \lim_{\substack{s \rightarrow t \\ s < t}} \nu_w^s(\varphi) \quad \forall \varphi \in \mathcal{C}(Y).$$

Proposition 79. Let $(f_n)_{n \in \mathbb{N}}$ be an increasing sequence of functions on Y_n , $f_n \geq 0$, $f_n \in \mathcal{Y}$ and J -integrable $\forall n \in \mathbb{N}$, such that f_n increases everywhere to a J -integrable function f . Further, let $\forall \lambda w$, $\forall n$, $\forall t$, **92**

$$\nu_w^{t-}(f_n) = \lim_{\substack{s \rightarrow t \\ s < t}} \nu_w^s(f_n). \text{ Then, } \forall \lambda w, \forall t, f \text{ is } \nu_w^{t-} \text{ integrable and } \forall \lambda w, \forall t,$$

$$\nu_w^{t-}(f) = \lim_{\substack{s \rightarrow t \\ s < t}} \nu_w^s(f)$$

Proof. By passing to a subsequence if necessary, we shall assume that $\forall n \in \mathbb{N}$,

$$\int |f_n - f| dJ \leq \frac{1}{2^{2n}}.$$

Let $h = \sum_{n=1}^{\infty} 2^n |f_n - f|$. Then $h \geq 0$, $h \in \mathcal{Y}$ and J -integrable.

$$\forall w, \forall t, |\nu_w^t(f_n) - \nu_w^t(f)| \leq \frac{1}{2^n} \nu_w^t(h).$$

By proposition (4, § 3, 72), $\forall \lambda w$, $\nu_w(h)$ is a right continuous, regulated function on \mathbb{R} and hence is locally bounded. Hence $\forall \lambda w$, $\nu_w^t(f_n)$ converges to $\nu_w^t(f)$ as $n \rightarrow \infty$ locally uniformly in the variable t . Hence,

$$\forall \lambda w, \forall t \in \mathbb{R}, \lim_{n \rightarrow \infty} \lim_{\substack{s \rightarrow t \\ s < t}} \nu_w^s(f_n) = \lim_{\substack{s \rightarrow t \\ s < t}} \nu_w^s(f).$$

(Proposition (4, § 3, 72) guarantees the existence of $\lim_{\substack{s \rightarrow t \\ s < t}} \nu_w^s(f_n)$ and $\lim_{\substack{s \rightarrow t \\ s < t}} \nu_w^s(f)$, $\forall n, \forall t$). Hence,

$$\forall \lambda w, \forall t \in \mathbb{R}, \lim_{n \rightarrow \infty} \nu_w^{t-}(f_n) = \lim_{\substack{s \rightarrow t \\ s < t}} \nu_w^s(f).$$

Now, since f is J -integrable, by proposition (4, § 3, 72), $\forall \lambda w$, $\nu_w(f)$ is a right continuous regulated function on \mathbb{R} and hence

$$\forall \lambda w, \forall t, \lim_{\substack{s \rightarrow t \\ s < t}} \nu_w^s(f) < +\infty.$$

Therefore, $\forall \lambda w, \forall t \in \mathbb{R}$ $\lim_{n \rightarrow \infty}$. But $\forall w \in \Omega, \forall t \in \mathbb{R}, \lim_{n \rightarrow \infty} \nu_w^{t-}(f_n) = \nu_w^{t-}(f)$ since $f_n \uparrow f$ everywhere on Y . Hence,

$$\forall \lambda w, \forall t \in \mathbb{R}, \nu_w^{t-}(f) < +\infty$$

and

$$\forall \lambda w, \forall t \in \mathbb{R}, \nu_w^{t-}(f) = \lim_{\substack{s \rightarrow t \\ s < t}} \nu_w^s(f).$$

□

Theorem 80. $\forall B \in \mathcal{B}, \forall \lambda w, \forall t$

$$\nu_w^{t-}(\chi_B) = \lim_{\substack{s \rightarrow t \\ s < t}} \nu_w^s(\chi_B).$$

Proof. Note that $\forall B \in \mathcal{B}, \forall \lambda w, \forall t, \lim_{\substack{s \rightarrow t \\ s < t}} \nu_w^s(\chi_B)$ exists and is finite, since $\nu(\chi_B)$ is a regular supermartingale.

Let U be an open set, $U \neq \emptyset$. Then χ_U is lower semi continuous and J -integrable since J is a finite measure. Since Y is a metrizable space, \exists an increasing sequence $(\varphi_n)_{n \in \mathbb{N}}$ of real valued non-negative continuous functions on Y such that $\varphi_n(y) \uparrow \chi_U(y)$ for all $y \in Y$.

$$\forall \lambda w, \forall t, \forall n, \nu_w^{t-}(\varphi_n) = \lim_{\substack{s \rightarrow t \\ s < t}} \nu_w^s(\varphi_n).$$

Hence, by the previous proposition (5, § 3, 79), $\forall \lambda w, \forall t, \nu_w^{t-}(\chi_U) = \lim_{\substack{s \rightarrow t \\ s < t}} \nu_w^s(\chi_U)$. Let

$$\mathcal{C} = \{C \in \mathcal{B} \mid \nu_w^{t-}(\chi_C) = \lim_{\substack{s \rightarrow t \\ s < t}} \nu_w^s(\chi_C)\}.$$

This class is a d -system, again by the previous proposition (5, § 3, 79). This d -system contains the π -system \mathcal{U} of all open subsets of Y . Hence, \mathcal{C} contains the σ -algebra \mathcal{B} , which is generated by \mathcal{U} . Hence,

$$\forall B \in \mathcal{B}, \nu_w^{t-}(\chi_B) = \lim_{\substack{s \rightarrow t \\ s < t}} \nu_w^s(\chi_B).$$

□

Proposition 81. Let f be any J -integrable extended real valued function on $Y, f \in \mathcal{B}$. Then, $\forall \lambda w, \forall t, f$ is ν_w^{t-} -integrable and

$$\forall \lambda w, \forall t, \nu_w^{t-}(f) = \lim_{\substack{t' \rightarrow t \\ t' < t}} \nu_w^{t'}(f).$$

Proof. Note that the existence of $\lim_{\substack{t' \rightarrow t \\ t' < t}} v_w^{t'}(f) \quad \forall t \in \mathbb{R}$ is guaranteed by

Corollary (4, § 3, 73).

Sufficient to prove when f is ≥ 0 , $f \in \mathcal{Y}$ and J -integrable.

If s is any step function on Y , $0 \leq s \leq f$ and $s \in \mathcal{Y}$, then s is J -integrable and it follows from the previous theorem (5, § 3, 80), that

$$\forall \lambda w, \forall t \in \mathbb{R}, v_w^{t-}(s) = \lim_{\substack{t' \rightarrow t \\ t' < t}} v_w^{t'}(s).$$

Now, we can find an increasing sequence $(s_n)_{n \in \mathbb{N}}$ of step functions on Y , $s_n \in \mathcal{Y}$ and $0 \leq s_n \leq f \quad \forall n \in \mathbb{N}$ such that $\forall y \in Y, s_n(y) \uparrow f(y)$.

Now, from proposition (5, § 3, 79) it follows that $\forall \lambda w, \forall t, f$ is v_w^{t-} -integrable and

$$\forall \lambda w, \forall t, v_w^{t-}(f) = \lim_{\substack{t' \rightarrow t \\ t' < t}} v_w^{t'}(f).$$

□

Proposition 82. *Let E be a Banach space over \mathbb{R} . Let g be a step function on Y , with values in E , $g \in \mathcal{Y}$ and J -integrable. Then, $\forall \lambda w, \forall t, g$ is v_w^{t-} -integrable and*

$$\forall \lambda w, \forall t, v_w^{t-}(g) = \lim_{\substack{t' \rightarrow t \\ t' < t}} v_w^{t'}(g).$$

Proof. Note that the existence of $\lim_{\substack{t' \rightarrow t \\ t' < t}} v_w^{t'}(g) \forall t$, is guaranteed by proposition (4, § 3, 74).

Let g be of the form $\sum_{i=1}^n \chi_{A_i} x_i$ where $\forall i = 1, \dots, n, A_i \in \mathcal{Y}, x_i \in E$ and $A_i \cap A_j = \emptyset$ if $i \neq j$.

Since $\forall w \in \Omega, \forall t \in \mathbb{R}, v_w^{t-}$ is a finite measure on $\mathcal{Y}, \forall i, \forall w \in \Omega, \forall t \in \mathbb{R}, v_w^{t-}(A_i) < +\infty$. Hence $\forall w \in \Omega, \forall t \in \mathbb{R}, g$ is v_w^{t-} -integrable, and $\forall w \in \Omega, \forall t \in \mathbb{R}, v_w^{t-}(A_i) < +\infty$. Hence $\forall w \in \Omega, \forall t \in \mathbb{R}, g$ is v_w^{t-} -integrable, and $\forall w \in \Omega, \forall t \in \mathbb{R},$

$$v_w^{t-}(g) = \sum_{i=1}^n v_w^{t-}(A_i) x_i.$$

By theorem (5, § 3, 80),

$$\forall \lambda w, \forall t, \forall i, \nu_w^{t-}(A_i) = \lim_{\substack{t' \rightarrow t \\ t' < t}} \nu_w^{t'}(A_i).$$

Hence $\forall \lambda w, \forall t$,

$$\begin{aligned} \nu_w^{t-}(g) &= \sum_{i=1}^n \lim_{\substack{t' \rightarrow t \\ t' < t}} \nu_w^{t'}(A_i) x_i \\ &= \lim_{\substack{t' \rightarrow t \\ t' < t}} \lim_{i=1}^n \nu_w^{t'}(A_i) x_i \\ &= \lim_{\substack{t' \rightarrow t \\ t' < t}} \nu_w^{t'}(g) \end{aligned}$$

□

Theorem 83. Let f be a J -integrable function on Y , with values in a Banach space E over \mathbb{R} , $f \in \mathcal{Y}$. Then $\forall \lambda w, \forall t$, f is ν_w^{t-} -integrable and

$$\forall \lambda w, \forall t, \nu_w^{t-}(f) = \lim_{\substack{t' \rightarrow t \\ t' < t}} \nu_w^{t'}(f).$$

Proof. Note that the existence of $\lim_{\substack{t' \rightarrow t \\ t' < t}} \nu_w^{t'}(f)$, $\forall \lambda w$, for every t is guaranteed by proposition (4, § 3, 75).

There exists a sequence $(g_n)_{n \in \mathbb{N}}$ of step functions on Y with values in E , $g_n \in \mathcal{Y} \forall n \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N}, \int |g_n - f| dJ \leq \frac{1}{2^{2n}}.$$

Let $h = \sum_{n=1}^{\infty} 2^n |g_n - f|$. Then $h \geq 0$, $h \in \mathcal{Y}$ and is J -integrable. 95

$$\forall n, |f| \leq |g_n - f| + |g_n| \leq \frac{h}{2^n} + |g_n|.$$

By proposition (5, § 3, 81), $\forall \lambda w, \forall t$, h is ν_w^{t-} -integrable, and by the previous proposition (5, § 3, 82), $\forall w, \forall t, \forall n$, g_n is ν_w^{t-} -integrable. Hence, $\forall \lambda w, \forall t$, f is ν_w^{t-} -integrable. Moreover, $\forall \lambda w, \forall t$,

$$| \nu_w^{t-}(g_n) - \nu_w^{t-}(f) | = | \nu_w^{t-}(g_n - f) |$$

$$\leq \frac{1}{2^n} v_w^{t^-}(h).$$

Since $\forall \lambda w, \forall t, v_w^{t^-}(h) < +\infty$, it follows that $\lim_{n \rightarrow \infty} v_w^{t^-}(g_n)$ exists and is equal to $v_w^{t^-}(f)$. (1)

$$\forall \lambda w, \forall t, |v_w^t(g_n) - v_w^t(f)| \leq v_w^t(|g_n - f|) \leq \frac{1}{2^n} v_w^t(h).$$

By proposition (4, § 3, 72), $\forall \lambda w, v_w(h)$ is right continuous and regulated. Hence, $\forall \lambda w, v_w(h)$ is a locally bounded function on \mathbb{R} .

Hence, $\forall \lambda w, v_w^t(g_n)$ converges to $v_w^t(f)$ in E as $n \rightarrow \infty$ locally uniformly in the variable t . Hence

$$\lim_{n \rightarrow \infty} \lim_{\substack{t' \rightarrow t \\ t' < t}} v_w^{t'}(g_n) = \lim_{\substack{t' \rightarrow \infty \\ t' < t}} v_w^{t'}(f).$$

Therefore,

$$\lim_{n \rightarrow \infty} v_w^{t^-}(g_n) = \lim_{\substack{t' \rightarrow t \\ t' < t}} v_w^{t'}(f) \quad (2)$$

From (1) and (2), we see that

$$\forall \lambda w, \forall t, v_w^{t^-}(f) = \lim_{\substack{t' \rightarrow t \\ t' < t}} v_w^{t'}(f).$$

□

Chapter 6

Regular Disintegrations

1 Basic Definitions

Throughout this chapter, let us assume that $(\Omega, \mathcal{O}, \lambda)$ is a measure space, $(\mathcal{C}^t)_{t \in \mathbb{R}}$ is an increasing right continuous family of sub σ -algebras of $\hat{\mathcal{O}}_\lambda$, λ restricted to \mathcal{C}^t is σ -finite $\forall t \in \mathbb{R}$ and that $\forall t$, λ has a disintegration $(\lambda_w^t)_{w \in \Omega}$ with respect to \mathcal{C}^t . 96

Then $(w, t) \rightarrow \lambda_w^t$ is a measure valued martingale on $\Omega \times \mathbb{R}$ with values in $\mathfrak{m}^+(\Omega, \mathcal{O})$, adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$.

Definition 84. $(\lambda_w^t)_{w \in \Omega}$ is said to be a regular disintegration of λ with respect to the family $(\mathcal{C}^t)_{t \in \mathbb{R}}$ if $(w, t) \rightarrow \lambda_w^t$ is a regular martingale adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$.

From the theorem (3, § 6, 44), we see that if Ω is a topological space, \mathcal{O} is its Borel σ -algebra and Ω has the λ -compactness metrizable property, then $\forall t \in \mathbb{R}$, λ has a disintegration $(\delta_w^t)_{w \in \Omega}$ with respect to \mathcal{C}^t . $(w, t) \rightarrow \delta_w^t$ is a measure valued martingale adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$. If further, λ is finite, then by theorem (5, § 2, 78), this measure valued martingale has a regular modification, say $(\lambda_w^t)_{w \in \Omega, t \in \mathbb{R}}$. Then, $(\lambda_w^t)_{w \in \Omega, t \in \mathbb{R}}$ is a regular disintegration of λ with respect to $(\mathcal{C}^t)_{t \in \mathbb{R}}$. Hence, if we assume that Ω is a topological space, \mathcal{O} is its Borel σ -algebra, Ω has the λ -compactness metrizable property, $(\mathcal{C}^t)_{t \in \mathbb{R}}$ is an increasing right continuous family of σ -algebras contained in $\hat{\mathcal{O}}_\lambda$ and λ is a finite measure

on \mathcal{O} , then λ has a regular disintegration with respect to $(\mathcal{C}^t)_{t \in \mathbb{R}}$.

Definition 85. A set $F \subset \Omega \times \mathbb{R}$ is said to be λ -evanescent if its projection on Ω is of λ -measure zero.

97 Note that if F is a λ -evanescent set, then there exists a set $B \in \hat{\mathcal{O}}_\lambda$ such that $\lambda(B) = 0$ and $F \subset B \times \mathbb{R}$. Hence, if a property $P(w, t)$ holds except for a λ -evanescent set, then there exists a set $B \in \hat{\mathcal{O}}_\lambda$ with $\lambda(B) = 0$ such that if $w \notin B$, $P(w, t)$ holds for all $t \in \mathbb{R}$. Conversely, if there exists a set $B \in \hat{\mathcal{O}}_\lambda$ with $\lambda(B) = 0$, such that a property $P(w, t)$ holds for all t , whenever $w \notin B$, then the property holds except for the λ -evanescent set $B \times \mathbb{R}$.

2 Properties of regular disintegrations

Let us assume hereafter that $(\lambda_w^t)_{\substack{w \in \mathbb{R} \\ t \in \mathbb{R}}}$ is a regular disintegration of λ with respect to $(\mathcal{C}^t)_{t \in \mathbb{R}}$.

Proposition 86. Let $B \in \mathcal{O}$ with $\lambda(B) = 0$. Then $\forall_{\lambda} w, \forall t, \lambda_w^t(B) = 0$.

Proof. We have $\forall t, \forall_{\lambda} w, \lambda_w^t(B) = 0$. Hence,

$$\forall_{\lambda} w, \forall t \in \mathbb{Q}, \lambda_w^t(B) = 0.$$

Since $\forall_{\lambda} w, t \rightarrow \lambda_w^t(B)$ is right continuous, it follows that

$$\forall_{\lambda} w, \forall t \in \mathbb{R}, \lambda_w^t(B) = 0.$$

□

Proposition 87. λ_w^t is a probability measure except for a λ -evanescent set.

Proof. $\forall t, \forall_{\lambda} w, \lambda_w^t$ is a probability measure. Therefore, $\forall_{\lambda} w, \forall t \in \mathbb{Q}, \lambda_w^t$ is a probability measure.

Since $\forall_{\lambda} w, t \rightarrow \lambda_w^t(\Omega)$ is right continuous, it follows that $\forall_{\lambda} w, \forall t \in \mathbb{R}, \lambda_w^t$ is a probability measure. □

Proposition 88. *Let $A \in \mathcal{C}^t \forall t \in \mathbb{R}$. Then $\forall \lambda w, \forall t, \lambda_w^t$ is carried by A or by $\mathbb{C}A$, according as $w \in A$ or $w \in \mathbb{C}A$.*

98 *Proof.* By proposition (3, § 7, 49), we know that $\forall t, \forall B \in \mathcal{C}^t, \forall \lambda w \lambda_w^t$ is carried by B or by $\mathbb{C}B$ according as $w \in B$ or $w \in \mathbb{C}B$. i.e. $\forall t, \forall B \in \mathcal{C}^t, \forall \lambda w, \chi_B(w) \cdot \lambda_w^t(\mathbb{C}B)$ and $\chi_{\mathbb{C}B}(w) \cdot \lambda_w^t(B)$ are both zero.

Since $A \in \mathcal{C}^t$ for all t , we have therefore, $\forall \lambda w, \forall t \in \mathbb{Q} \chi_A(w) \cdot \lambda_w^t(\mathbb{C}A)$ and $\chi_{\mathbb{C}A}(w) \cdot \lambda_w^t(A)$ are both zero.

Since $\forall \lambda w, t \rightarrow \lambda_w^t(\mathbb{C}A)$ and $t \rightarrow \lambda_w^t(A)$ are right continuous, it follows that $\forall \lambda w, \forall t \in \mathbb{R}, \chi_B(w) \cdot \lambda_w^t(\mathbb{C}A)$ and $\chi_{\mathbb{C}A}(w) \lambda_w^t(A)$ are both zero. \square

Proposition 89. *Let f be a bounded regular supermartingale, adapted to the family $(\mathcal{C}^t)_{t \in \mathbb{R}}$. Then, $\forall \lambda w, \forall s, t \rightarrow \lambda_w^B(f^t)$ is right continuous.*

Proof. Since f is a regular supermartingale, $\forall \lambda w, f_w$ is right continuous, i.e. $\exists B \in \mathcal{O}$ with $\lambda(B) = 0$ such that if $w \notin B, t \rightarrow f^t(w)$ is right continuous.

Let t be fixed. Let $t_n \downarrow t$. Then, if $w \notin B, f^{t_n}(w) \rightarrow f^t(w)$.

By proposition (6, § 2, 86),

$$\forall \lambda w, \forall s \lambda_w^s(B) = 0.$$

Hence, $\forall \lambda w, \forall s, \forall \lambda_w^s w', f^{t_n}(w') \rightarrow f^t(w')$. Since by proposition (6, § 2, 87), $\forall \lambda w, \forall s, \lambda_w^s$ is a probability measure, and since f is bounded, it follows by the dominated convergence theorem, that

$$\forall \lambda w, \forall s, \lambda_w^s(f^{t_n}) \rightarrow \lambda_w^s(f^t).$$

\square

Proposition 90. *Let f be a bounded regular supermartingale adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$. Then, $\forall \lambda w, \forall s, t \rightarrow \lambda_w^s(f^t)$ is a decreasing function in $[s, +\infty)$.*

Proof. Let $\{t, t'\}$ be a fixed pair of real numbers, $t < t'$. Since f is a supermartingale adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$, we have

$$\forall s, s \leq t, \forall \lambda w, \lambda_w^s(f^{t'}) \leq \lambda_w^s(f^t).$$

Therefore, $\forall \lambda w, \forall s \in \mathbb{Q} \ s \leq t, \lambda_w^s(f^{t'}) \leq \lambda_w^s(f^t)$.

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Now, $(w, s) \rightarrow \lambda_w^s(f^{t'})$ and $(w, s) \rightarrow \lambda_w^s(f^t)$ are regular supermartingales. Hence, $\forall \lambda w, s \rightarrow \lambda_w^s(f^{t'})$ and $s \rightarrow \lambda_w^s(f^t)$ are right continuous. Therefore $\forall \lambda w, \forall s < t, \lambda_w^s(f^{t'}) \leq \lambda_w^s(f^t)$.

Now, $\forall \lambda w, \lambda_w^t(f^t) = f^t(w)$. Since f is a supermartingale,

$$\forall \lambda w, \lambda_w^t(f^{t'}) \leq f^t(w).$$

Hence, $\forall \lambda w, \lambda_w^t(f^{t'}) \leq \lambda_w^t(f^t)$. Therefore,

$$\forall \lambda w, \forall s \leq t, \lambda_w^s(f^{t'}) \leq \lambda_w^s(f^t).$$

Thus, for every pair $\{t, t'\} t < t', \forall \lambda w, \forall s \leq t, \lambda_w^s(f^{t'}) \leq \lambda_w^s(f^t)$. Hence, $\forall \lambda w, \forall$ pair $\{t, t'\}, t \in \mathbb{Q} \ t' \in \mathbb{Q} \ t < t',$

$$\forall s \leq t, \lambda_w^s(f^{t'}) \leq \lambda_w^s(f^t).$$

By the previous proposition (6, § 2, 89), $\forall \lambda w, \forall s, t \rightarrow \lambda_w^s(f^t)$ is right continuous. Therefore, $\forall \lambda w, \forall$ pair $\{t, t'\}, t, t', \forall s \leq t,$

$$\lambda_w^s(f^{t'}) \leq \lambda_w^s(f^t).$$

Thus, $\forall \lambda w, \forall s, t \rightarrow \lambda_w^s(f^t)$ is a decreasing function in $[s, +\infty)$. \square

Theorem 91. *Let f be a regular supermartingale adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$. Then $\forall \lambda w, \forall s, t \rightarrow \lambda_w^s(f^t)$ is decreasing and right continuous in $[s, +\infty)$ and*

$$\forall \lambda w, \forall s, \lambda_w^s(f^s) = f^s(w).$$

Proof. If f is a bounded regular supermartingale, then from the previous propositions, proposition (6, § 2, 89) and proposition (6, § 2, 90), we see that

$$\forall \lambda w, \forall s, t \rightarrow \lambda_w^s(f^t)$$

100 is a decreasing right continuous function in $[s, +\infty)$. Hence, $\forall \lambda w, \forall s, t \rightarrow \lambda_w^s(f^t)$ is decreasing and lower semi-continuous in $[s, +\infty)$. (At s , we mean only right lower semi-continuity).

If f is an arbitrary regular supermartingale, not necessarily bounded, $\forall m \in \mathbb{N}, \inf(f, m)$ is a bounded regular supermartingale. Hence,

$\forall m \in \mathbb{N}, \forall \lambda w, \forall s, t \rightarrow \lambda_w^s[\{\inf(f, m)\}^t]$ is decreasing and lower semi-continuous in $[s, +\infty)$.

Since $\forall t, \{\inf(f, m)\}^t \uparrow f^t$, we have

$$\forall s, \forall t, \forall w, \lambda_w^s(f^t) = \lim_{m \rightarrow \infty} \lambda_w^s[\{\inf(f, m)\}^t].$$

Therefore, $\forall \lambda w, \forall s, t \rightarrow \lambda_w^s(f^t)$ is decreasing and lower semi-continuous in $[s, +\infty)$.

If a function, in an interval is decreasing and lower semi-continuous, if is right continuous there.

Hence, $\forall \lambda w, \forall s, t \rightarrow \lambda_w^s(f^t)$ is decreasing and right continuous in $[s, +\infty)$.

Let us now prove that

$$\forall \lambda w, \forall s, \lambda_w^s(f^s) = f^s(w).$$

We have $\forall s, \forall \lambda w, \lambda_w^s(f^s) = f^s(w)$. Hence,

$$\forall \lambda w, \forall s \in \mathbb{Q}, \lambda_w^s(f^s) = f^s(w).$$

Since $\forall \lambda w, s \rightarrow f^s(w)$ is right continuous, to prove the theorem, it is sufficient to prove that $\forall \lambda w, s \rightarrow \lambda_w^s(f^w)$ is right continuous. Now,

$$\mathbb{R} = \bigcup_{\substack{k \in \mathbb{Z} \\ n \in \mathbb{N}}} \left[\frac{k}{2^n}, \frac{k+1}{2^n} \right).$$

Define $\forall n \in \mathbb{N}, w \in \Omega$

$$f_n^s(w) = \lambda_w^s(f^{\frac{k+1}{2^n}}) \text{ if } s \in \left[\frac{k}{2^n}, \frac{k+1}{2^n} \right) ..$$

We claim that $\forall n, f_n$ is a regular supermartingale adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$. **101**
 $\forall n$, the regularity of f_n is clear, since $(\lambda_w^s)_{w \in \Omega, s \in \mathbb{R}}$ is a regular disintegration of λ . We have to only prove that $\forall n \in \mathbb{N}, f_n$ is a supermartingale adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$.

Clearly, f_n is adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$, $\forall n \in \mathbb{N}$. We have to only prove that $\forall n, \forall t, t' \in \mathbb{R}, t < t', \forall A \in \mathcal{C}^t$,

$$\int_A f_n^{t'}(w) d\lambda(w) \leq \int_A f_n^t(w) d\lambda(w).$$

This is clear, if both t and t' belong to the same interval $\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$ for some $t \in \mathbb{Z}$. Actually, in this case, we have even equality as $(s, w) \rightarrow \lambda_w^s(f \frac{k+1}{2^n})$ is a martingale.

It is sufficient to prove the above inequality when t and t' belong to two consecutive intervals, any $t \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$ and $t' \in \left[\frac{k+1}{2^n}, \frac{k+2}{2^n}\right)$, because for other values of t and t' , $t < t'$, the above inequality will then follow from the transitivity of the conditional expectations.

Again to prove the inequality, it is sufficient to prove it when $t = k/2^n$ for some $k \in \mathbb{Z}$ and $t' = \frac{k+1}{2^n}$, because of the transitivity of the conditional expectations and because of the fact that $(s, w) \rightarrow \lambda_w^s(f \frac{k+1}{2^n})$ is a martingale; i.e. we have to only prove that $\forall n \in \mathbb{N}, \forall k \in \mathbb{Z}, \forall A \in \mathcal{C}^{k/2^n}$

$$\int_A \lambda_w^{\frac{k+1}{2^n}}(f \frac{k+2}{2^n}) d\lambda(w) \leq \int_A \lambda_w^{k/2^n}(f \frac{k+1}{2^n}) d\lambda(w).$$

But this follows immediately from the fact that f is a supermartingale. It is easy to check that

$$\forall s, \forall n, \quad \forall_\lambda w, \quad f_n^s(w) \leq f_{n+1}^s(w).$$

Therefore, since $\forall n \in \mathbb{N}$, f_n is regular, we have

$$\forall_\lambda w, \forall n, \forall s, \quad f_n^s(w) \leq f_{n+l}^s(w).$$

102 Hence $\forall_\lambda w, \forall s, \lim_{n \rightarrow \infty} f_n^s(w)$ exists. But

$$\forall \lambda w, \forall s, t \rightarrow \lambda_w^s(f^t)$$

is right continuous in $[s, +\infty)$. Hence

$$\forall \lambda w, \forall s, \lim_{n \rightarrow \infty} f_n^s(w) = \lambda_w^s(f^s).$$

Since $(f_n)_{n \in \mathbb{N}}$ is an increasing sequence of regular supermartingales, it follows from the ‘Upper envelope theorem’ that

$$\forall \lambda w, s \rightarrow \lambda_w^s(f^s)$$

is right continuous. □

We now state and prove another important theorem, which we call the “theorem of trajectories”. In the course of the proof of this theorem, we need the concepts relating to “Well-measurable processes” and “ \mathcal{C} -analytic sets” where \mathcal{C} is a σ -algebra on Ω . We state about well measurable processes what we need. For the definition and properties of \mathcal{C} -analytic sets, we refer the reader to P.A. Meyer [1], Chapter, 3 and section 1 “Compact pavings and Analytic sets”.

By a *stochastic process* X on Ω , with values in a topological spaces E , we mean a collection $(X^t)_{t \in \mathbb{R}}$ of mappings on Ω with values in E such that $\forall t, X^t \in \hat{\mathcal{O}}_\lambda$. A stochastic process can be considered as a mapping X from $\Omega \times \mathbb{R}$ to E such that $\forall t \in \mathbb{R}, X^t \in \hat{\mathcal{O}}_\lambda$. A stochastic process X is said to be *adapted* to $(\mathcal{C}^t)_{t \in \mathbb{R}}$ if $\forall t \in \mathbb{R}, X^t \in \mathcal{C}^t$.

Definition 92. A stochastic process X on Ω with values in \mathbb{R} is said to be well-measurable if it belongs to the σ -algebra on $\Omega \times \mathbb{R}$, generated by all sets $A \subset \Omega \times \mathbb{R}$ such that $(w, t) \rightarrow \chi_A(w, t)$ is regulated and right continuous and is adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$. 103

See definition D14 in page 156 and remark (b) in page 157 in P.A. Meyer [1].

We need the following important fact.

If X is a right continuous adapted process on Ω with values in \mathbb{R} , then X is well-measurable.

For a proof see remark (c) in page 157, in P.A. Meyer [1].

Theorem 93 (The theorem of trajectories). *Let X be a right continuous stochastic process on Ω , adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$, with values in a topological space E which has a countable family of real valued continuous functions separating its points (for example, a completely regular Suslin space). Then,*

$$\forall \lambda w, \forall t \in \mathbb{R} \forall \lambda_w^t w', \forall s \leq t, X^s(w') = X^s(w).$$

i.e. $\forall \lambda w, \forall t, \lambda_w^t$ is carried by the set of all w' whose trajectories coincide with those of w upto the time t .

Proof. If \mathcal{B}_E is the Borel σ -algebra of E , it is easy to see that \mathcal{B}_E is countably separating. Let s be any fixed real number. Let h be a function on Ω with values in E such that $h \in \mathcal{C}^t$ for all $t \geq s$. Then proceeding as in the proof of proposition (3, § 7, 54) and proposition (6, § 2, 88), we can prove that

$$\forall \lambda w, \forall t \geq s, \forall \lambda_w^t w', h(w') = h(w).$$

Now $X^s \in \mathcal{C}^t \forall t \geq s$. Hence,

$$\forall s, \forall \lambda w, \forall t \geq s, \forall \lambda_w^t w', X^s(w') = X^s(w).$$

104 Therefore, $\forall \lambda w, \forall s \in \mathbb{Q}, \forall t \geq s, \forall \lambda_w^t w', X^s(w') = X^s(w)$. Hence, $\forall \lambda w, \forall t, \forall \lambda_w^t w', \forall s \in \mathbb{Q} s \leq t, X^s(w') = X^s(w)$,

Now, $\forall \lambda w, s \rightarrow X^s(w)$ is right continuous and hence $\forall \lambda w, \forall t, \forall \lambda_w^t w', s \rightarrow X^s(w')$ is right continuous. Hence

$$\forall \lambda w, \forall t, \forall \lambda_w^t w', \forall s < t, X^s(w') = X^s(w). \quad (1)$$

To prove the theorem, we have to only prove that $\forall \lambda w, \forall t, \forall \lambda_w^t w', X^t(w') = X^t(w)$.

First let us consider the case when $E = \mathbb{R}$.

Let us assume that X is a regular supermartingale adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$. Let M be any real number > 0 . Let $X_M = \inf(X, M)$. Then, X_M is also a regular supermartingale. Therefore, by theorem (6, § 2, 91),

$$\forall \lambda w, \forall t, \lambda_w^t(X_M^t) = X_M^t(w).$$

Let w be such that λ_w^t is a probability measure and $\lambda_w^t(X_M^t) = X_M^t(w)$. If $X^t(w) \geq M$, then $X_M^t(w) = M$ and $X_M^t(w') \leq M$ for all $w' \in \Omega$. Since $\lambda_w^t(X_M^t) = X_M^t(w)$, it follows therefore that $\forall \lambda_w^t w', X_M^t(w') = M = X_M^t(w)$ and hence,

$$\forall \lambda_w^t w', X^t(w') \geq M.$$

Since $\forall \lambda w, \forall t, \lambda_w^t$ is a probability measure and since $\forall \lambda w, \forall t, \lambda_w^t(X_M^t) = X_M^t(w)$, we see that

$$\forall M \in \mathbb{R}, M > 0, \forall \lambda w, \forall t, \text{ if } X^t(w) \geq M, \text{ then}$$

$$\forall \lambda_w^t w', X^t(w') \geq M.$$

Hence, $\forall \lambda w, \forall t, \forall M \in \mathbb{Q}, M > 0$, if $X^t(w) \geq M$, then $\forall \lambda_w^t w', X^t(w') \geq M$. Therefore $\forall \lambda w, \forall t, \forall \lambda_w^t w', X^t(w') \geq X^t(w)$. But by theorem (6, § 2, 91),

$$\forall \lambda w, \forall t, \lambda_w^t(X^t) = X^t(w).$$

Hence, $\forall \lambda w, \forall t, \forall \lambda_w^t w', X^t(w') = X^t(w)$.

This proves the theorem, when X is a regular supermartingale. 105

Since an adapted right continuous decreasing process with values in \mathbb{R} , is a regular supermartingale, the theorem holds when X is such a process. Therefore also, when X is a bounded adapted right continuous increasing process. By passing to the limit, the theorem therefore also holds when X is an adapted increasing right continuous process.

Now, let X be any regulated right continuous process with values in \mathbb{R} .

Let $\forall w \in \Omega$ and $t \in \mathbb{R}$,

$$\sigma(w, t) = X(w, t) - X(w, t-).$$

σ is again an adapted process with values in \mathbb{R} . Let $\alpha > 0$.

Since $\forall \lambda w, X_w$ is a regulated function on \mathbb{R} , $\forall \lambda w$, the number of t 's in any relatively compact interval of \mathbb{R} for which $\sigma(w, t) \geq \alpha$ is finite.

Let $n \in \mathbb{N}$ and $w \in \Omega$ and $t \in \mathbb{R}$. Define $\overset{\alpha}{M}_n(w, t)$ as the number of τ 's, $\tau \in (-n, t]$ for which $\sigma(w, \tau) \geq \alpha$, if $t > -n$ and zero otherwise.

Then $\forall \lambda w, \forall n, \overset{\alpha}{M}_n(w, t)$ is an integer for all t and it is easy to see that $\forall w$, for which X_w is regulated, given any $t \in \mathbb{R}$, $\exists \delta > 0$ such that

$M_n(w, \cdot)$ is constant in $[t, t + \delta)$. Hence $\forall_{\lambda} w, t \rightarrow M_n(w, t)$ is trivially a right continuous function. Moreover, $\forall_{\lambda} w, \forall n, M_n(w', \cdot)$ is an increasing function of t .

Hence, $\forall n \in \mathbb{N}$, $\overset{\alpha}{M}_n$ is an increasing right continuous process on $\Omega \times \mathbb{R}$.

Let us show that it is adapted to the family $(\hat{\mathcal{C}}_{\lambda}^t)_{t \in \mathbb{R}}$; i.e. let us show that

$$\forall t \in \mathbb{R}, \forall n \in \mathbb{N}, w \rightarrow \overset{\alpha}{M}_n(w, t) \in \hat{\mathcal{C}}_{\lambda}^t.$$

To understand the ideas involved in this prove, the reader is referred to the proof of T52 in page 71 of P.A. Meyer [1], where similar ideas are used.

Any right continuous or a left continuous process Z adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$ is *progressively measurable* with respect to $(\mathcal{C}^t)_{t \in \mathbb{R}}$ and hence $\forall t$, the restriction of Z to $\Omega \times (-n, t]$ belongs to $\mathcal{C}^t \otimes \mathcal{B}(-n, t]$ where $\mathcal{B}(-n, t]$ is the Borel σ -algebra of $(-n, t]$. (see D45 in page 68 and T47 in page 70 of P.A. Meyer [1]). Note that $(w, t) \rightarrow X(w, t-)$ is a left continuous process and hence σ is progressively measurable with respect to $(\mathcal{C}^t)_{t \in \mathbb{R}}$.

Since $\overset{\alpha}{M}_n^t$ is an integer valued function, to prove that $\forall n \in \mathbb{N}, \overset{\alpha}{M}_n^t \in \hat{\mathcal{C}}_{\lambda}^t$, it is sufficient to show that $\forall m \in \mathbb{N}$, $\{w : M_n^t(w) \geq m\}$ is a \mathcal{C}^t -analytic set.

Let A be the subset of $\Omega \times \mathbb{R}^{m+1}$ consisting of points $(w, -n, s_1, s_2, \dots, s_m)$ where $w \in \Omega, -n < s_1 < s_2 \dots < s_m \leq t$ and

$$\sigma(w, s_1) \geq \alpha, \sigma(w, s_2) \geq \alpha, \dots, \sigma(w, s_m) \geq \alpha.$$

This set $A \in \mathcal{C}^t \otimes \mathcal{B}^{m+1}(-n, t]$ where $\mathcal{B}^{m+1}(-n, t]$ is the Borel σ -algebra of $\underbrace{(-n, t] \times (-n, t] \dots \times (-n, t]}_{m+1 \text{ times}}$.

This is because of the fact that σ is progressively measurable with respect to $(\mathcal{C}^t)_{t \in \mathbb{R}}$.

The projection of A on Ω is precisely the set $\{w : \overset{\alpha}{M}_n^t(w) \geq m\}$. Hence $\{w : M_n^t(w) \geq m\}$ is a \mathcal{C}^t -analytic set. Hence

$$\forall n, \forall t, \overset{\alpha}{M}_n^t \in \hat{\mathcal{C}}_{\lambda}^t.$$

Let $\forall t, \mathfrak{F}^t$ be the σ -algebra $\bigcap_{u>t} \hat{\mathcal{C}}_\lambda^u$. Then $(\mathfrak{F}^t)_{t \in \mathbb{R}}$ is an increasing right continuous family of σ -algebras on Ω , contained in $\hat{\mathcal{O}}_\lambda$ and $\forall n \in \mathbb{N}, \overset{\alpha}{M}_n$ is adapted to $(\mathfrak{F}^t)_{t \in \mathbb{R}}$. Since $\overset{\alpha}{M}_n$ is an increasing right continuous process adapted to $(\mathfrak{F}^t)_{t \in \mathbb{R}}$, the theorem is valid for this process.

Hence $\forall n \in \mathbb{N}, \forall \alpha > 0, \forall \lambda w, \forall t, \forall \lambda_w^t w', \forall s \leq t, \overset{\alpha}{M}_n(w', s) = \overset{\alpha}{M}_n(w, s)$.

This shows that $\forall n \in \mathbb{N}, \forall \alpha > 0, \forall \lambda w, \forall t, \forall \lambda_w^t w'$, for all $s \leq t$, the number of jumps in $(-n, s]$ of magnitude greater than or equal to α are the same for X_w and $X_{w'}$.

Since this is true for all rational $\alpha > 0$ and the same thing analogously for all rational $\alpha < 0$, it follows that $\forall n \in \mathbb{N}, \forall \lambda w, \forall t, \forall \lambda_w^t w'$, for all $s \leq t$, the number of jumps in $(-n, s]$ of any given magnitude are the same for X_w and $X_{w'}$. 107

Hence $\forall n \in \mathbb{N}, \forall \lambda w, \forall t, \forall \lambda_w^t w'$, for all $s \leq t, X_{w'}$ and X_w have jumps precisely at the same points in $(-n, s]$ and the magnitudes of the jumps at each point of a jump are the same for $X_{w'}$ and X_w .

Since this is true for every $n \in \mathbb{N}$, we therefore have that

$$\forall \lambda w, \forall t, \forall \lambda_w^t w', X_{w'}$$

has a jump at a point $s \leq t$ if and only if X_w has one at s and the magnitudes of the jumps for X_w and $X_{w'}$ at s are the same.

Let $C = \{w \in \Omega \mid \forall t, \forall \lambda_w^t w', w' \text{ has a jump at a point } s \leq t \text{ if and only if } X_w \text{ has one at } s \text{ and the magnitudes of the jumps for } X_{w'} \text{ and } X_w \text{ at } s \text{ are the same}\}$. Then, by what we have seen, λ is carried by C .

Let $B = \{w \in \Omega \mid \forall t, \forall \lambda_w^t w', \forall s < t, X^s(w') = X^s(w)\}$. Then, from (1), λ is carried by B .

Let $w \in B \cap C$. Let t be any point of \mathbb{R} . Let X_w be continuous at t . We have, since $w \in B$,

$$\forall \lambda_w^t w', \forall s < t, X^s(w') = X^s(w).$$

Hence, $\forall \lambda_w^t w', \lim_{\substack{s \rightarrow t \\ s < t}} X^s(w')$ exists and is equal to $X^t(w)$, since X_w is continuous at t .

Let us prove that

$$\forall_{\lambda_w^t} w', \lim_{\substack{s \rightarrow t \\ s < t}} X^s(w') = X^t(w').$$

108 If not, the set $B_w^t = \{w' \in \Omega \mid \sum_{\substack{s \rightarrow t \\ s < t}} X^s(w') \neq X^t(w')\}$ is of positive λ_w^t -measure. But, since $w \in C$ and since X_w is continuous at t and hence has no jump at t , on no set of positive λ_w^t -measure, all the w' can have a jump at t . Hence B_w^t must be of λ_w^t -measure zero. Hence,

$$\forall_{\lambda_w^t} w', \lim_{\substack{s \rightarrow t \\ s < t}} X^s(w') = X^t(w') = X^t(w).$$

Therefore, $\forall_{\lambda_w^t} w', X^t(w') = X^t(w)$.

Now, let $w \in B \cap C$ and t be a point at which X_w is discontinuous. Then X_w has a jump at t and since X_w is right continuous at t , the jump is equal to $X(w, t) - X(w, t-)$. Hence, since $w \in C$, $\forall_{\lambda_w^t} w', w'$ has also a jump at t and

$$X(w', t) - X(w', t-) = X(w, t) - X(w, t-).$$

But since $w \in B$,

$$\forall_{\lambda_w^t} w', \forall s < t, X^s(w') = X^s(w)$$

Hence

$$\forall_{\lambda_w^t} w', X(w', t-) = X(w, t-).$$

Therefore,

$$\forall_{\lambda_w^t} w', X(w', t) = X(w, t).$$

Since λ is carried by $B \cap C$ and since $t \in \mathbb{R}$ is arbitrary, we have

$$\forall_{\lambda} w, \forall t, \forall_{\lambda_w^t} w', X^t(w') = X^t(w).$$

This combined with (1) gives that $\forall_{\lambda} w, \forall t, \forall_{\lambda_w^t} w', \forall s \leq t, X^s(w') = X^s(w)$.

Hence the theorem is proved when X is an adapted, regulated, right continuous real valued process.

109 Let \mathcal{C}

$$= \{A \subset \Omega \times \mathbb{R} \mid \chi_A \text{ is adapted and the theorem is true for the process } \chi_A\}.$$

Then \mathcal{C} is a σ -algebra on $\Omega \times \mathbb{R}$.

\mathcal{C} contains by the preceding, all the sets A for which χ_A is an adapted regulated right continuous process. Hence \mathcal{C} contains all the well-measurable processes.

Since a right continuous, adapted process is well-measurable, the theorem is true for any right continuous adapted process with values in \mathbb{R} .

Now, if $(f_n)_{n \in \mathbb{N}}$ is a countable family of continuous functions on E , separating the points of E , and if X is a right continuous adapted process with values in E , then $\forall n \in \mathbb{N}$, $f_n \circ X$ is a real valued, adapted right continuous process and hence the theorem is true for $f_n \circ X$, $\forall n \in \mathbb{N}$. Since the f_n 's separate the points of E and are countable, the theorem is true for X . \square

Theorem 94. *Let \mathcal{O} be countably generated. Then,*

$$\forall \lambda w, \forall s, \forall t, \int \lambda_{w'}^t \lambda_w^s(dw') = \lambda_w^{\text{Min}(s,t)}.$$

Proof. $\forall B \in \mathcal{O}, \forall s, t, s \leq t,$

$$\forall \lambda w, \int \lambda_{w'}^t(B) \lambda_w^s(dw') = \lambda_w^s(B)$$

since $(t, w') \rightarrow \lambda_{w'}^t(B)$ is a martingale. Hence,

$$\forall B \in \mathcal{O}, \forall t, \forall \lambda w, \forall s \in \mathbb{Q}, s \leq t,$$

$$\int \lambda_{w'}^t(B) \lambda_w^s(dw') = \lambda_w^s(B).$$

Since $(\lambda_w^s)_{\substack{w \in \Omega \\ s \in \mathbb{R}}}$ is a regular disintegration, $\forall \lambda w, s \rightarrow \lambda_w^s(B)$ and $s \rightarrow \int \lambda_{w'}^t(B) \lambda_w^s(dw')$ are right continuous. Hence,

$$\forall B \in \mathcal{O}, \forall t, \forall \lambda w, \forall s < t,$$

$$\int \lambda_{w'}^t(B) \lambda_w^s(dw') = \lambda_w^s(B).$$

Therefore,

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$$\begin{aligned} \forall B \in \mathcal{O}, \forall \lambda w, \forall t \in \mathbb{Q}, \forall s < t, \\ \int \lambda_{w'}^t(B) \lambda_w^s(dw') = \lambda_w^s(B). \end{aligned}$$

Since $(\lambda_w^t)_{\substack{w \in \Omega \\ t \in \mathbb{R}}}$ is a regular disintegration, $\forall B \in \mathcal{O}, \forall \lambda w, t \rightarrow \lambda_w^t(B)$ is right continuous. Hence $\forall B \in \mathcal{O}, \forall \lambda w, \forall s, \forall \lambda_w^s w', t \rightarrow \lambda_{w'}^t(B)$ is right continuous. Hence,

$$\begin{aligned} \forall B \in \mathcal{O}, \forall \lambda w, \forall t, \forall s < t, \\ \int \lambda_{w'}^t(B) \lambda_w^s(dw') = \lambda_w^s(B) \end{aligned} \quad (1)$$

Since $(\lambda_w^s(B))_{\substack{w \in \Omega \\ s \in \mathbb{R}}}$ is an adapted right continuous process with values in \mathbb{R} , by the previous theorem (6, § 2, 93),

$$\forall B \in \mathcal{O}, \forall \lambda w, \forall s, \forall \lambda_w^s w', \forall t \leq s, \lambda_{w'}^t(B) = \lambda_w^t(B).$$

Therefore,

$$\forall \lambda w, \forall s, \forall t \leq s, \int \lambda_{w'}^t(B) \lambda_w^s(dw') = \lambda_w^t(B),$$

since $\forall \lambda w, \forall s, \lambda_w^s$ is a probability measure.

II

From (I) and (II), we see therefore that

$$\forall B \in \mathcal{O}, \forall \lambda w, \forall s, \forall t, \int \lambda_{w'}^t(B) \lambda_w^s(dw') = \lambda_w^{\text{Min}(s,t)}(B).$$

Since \mathcal{O} is countably generated, by the Monotone class theorem,

$$\forall \lambda w, \forall s, \forall t, \forall B \in \mathcal{O}, \int \lambda_{w'}^t(B) \lambda_w^s(dw') = \lambda_w^{\text{Min}(s,t)}(B).$$

Hence,

$$\forall \lambda w, \forall s, \forall t, \int \lambda_{w'}^t \lambda_w^s(dw') = \lambda_w^{\text{Min}(s,t)}.$$

□

Chapter 7

Strong σ -Algebras

1 A few propositions

We shall prove in this section two propositions which will be used later on. 111

Proposition 95. *Let $(\Omega, \mathcal{O}, \lambda)$ be a measure space and \mathcal{C} , a σ -algebra contained in $\hat{\mathcal{O}}_\lambda$. Let λ restricted to \mathcal{C} be σ -finite. Then, the following are equivalent:*

- (i) λ is disintegrated with respect to \mathcal{C} by the constant measure valued function $w \rightarrow \lambda_w^\mathcal{C} = \lambda$.
- (ii) λ is ergodic on \mathcal{C} i.e. $\forall A \in \mathcal{C}, \lambda(A) = 0$ or 1 .

Proof. Let us assume (i) and prove (ii).

By the definition of a measure space, $\lambda \neq 0$. (i) implies that λ is a probability measure. By proposition (3, § 7, 49), for any disintegration $(\lambda_w^\mathcal{C})_{w \in \Omega}$ of λ with respect to \mathcal{C} , given $A \in \mathcal{C}$, $\forall \lambda w$, $\lambda_w^\mathcal{C}$ is carried by A or by $\complement A$, according as $w \in A$ or $w \in \complement A$. Therefore, given $A \in \mathcal{C}$, either $\lambda(A) = 0$ or if $\lambda(A) > 0$, there exists a $w \in A$ such that $\lambda_w^\mathcal{C}$ is carried by A . Applying this to our special case where $\lambda_w^\mathcal{C} = \lambda \quad \forall w \in \Omega$, we see that if $\lambda(A) > 0$, λ is carried by A and hence $\lambda(A) = 1$, since λ is a probability measure. Thus, λ is ergodic on \mathcal{C} .

Let us now assume (ii) and prove (i). (ii) implies that λ is a probability measure.

To verify that λ is disintegrated with respect to \mathcal{C} by the constant measure valued function $w \rightarrow \lambda_w^\mathcal{C} = \lambda$, according to proposition (3, § 7, 49), it is sufficient to verify that

(1) for any $A \in \mathcal{O}$, $w \rightarrow \lambda(A)$ belongs to \mathcal{C} .

(2) $\lambda = \int \lambda d\lambda(w)$

(3) Given $A \in \mathcal{C}$, λ is carried by A or by $\mathbb{C}A$.

(1) and (2) are clear and (3) follows from the fact that λ is ergodic. \square

Proposition 96. *Let (X, \mathfrak{X}, μ) be a measure space. Let $x \rightarrow \nu_x$ be a measure valued function on X with values in a measurable space (Ω, \mathcal{O}) , $\nu \in \mathfrak{X}$. Let $\rho = \int \nu_x \mu(dx)$. Let \mathcal{C} be a σ -algebra contained in $\hat{\mathcal{O}}_\rho \cap \bigcap_{x \in X} \hat{\mathcal{O}}_{\nu_x}$. Let $\forall_{\mu, x}$, ν_x have $(\lambda_w^\mathcal{C})_{w \in \Omega}$ as disintegration with respect to \mathcal{C} . Then ρ also has $(\lambda_w^\mathcal{C})_{w \in \Omega}$ as disintegration with respect to \mathcal{C} .*

Proof. We have to only prove that

$$\forall A \in \mathcal{C}, \chi_A \cdot \rho = \int_A \lambda_w^\mathcal{C} d\rho(w).$$

Let $B \in \mathcal{O}$.

$$\chi_A \cdot \rho(B) = \rho(A \cap B) = \int \nu_x(A \cap B) d\mu(x).$$

$$\forall_{\mu, x}, \nu_x(A \cap B) = \int \lambda_w^\mathcal{C}(A \cap B) d\nu_x(w).$$

$\int \lambda_w^\mathcal{C}(A \cap B) d\nu_x(w) = \nu_x(\lambda_w^\mathcal{C} \cdot (A \cap B)) = \nu_x(\chi_A \cdot \lambda_w^\mathcal{C} \cdot (B))$. Therefore,

$$\begin{aligned} \int \nu_x(A \cap B) d\mu(x) &= \int \nu_x(\chi_A \cdot \lambda_w^\mathcal{C} \cdot (B)) \mu(dx) \\ &= \int \chi_A(w) \cdot \lambda_w^\mathcal{C}(B) \rho(dw) \end{aligned}$$

$$= \left(\int_A \lambda_A^{\mathcal{C}} \rho(dw) \right)(B).$$

Since $B \in \mathcal{O}$ is arbitrary,

$$\chi_A \cdot \rho = \int_A \lambda_w^{\mathcal{C}} \rho(dw).$$

□

2 Strong σ -Algebras

Hereafter throughout this chapter, we shall make the following assumptions regarding Ω , \mathcal{O} and λ . 113

- (1) Ω is a topological space and \mathcal{O} is its Borel σ -algebra
- (2) \mathcal{O} is countably generated
- (3) λ is a probability measure on \mathcal{O} and
- (4) Ω has the λ -compactness metrizable property.

For example if Ω is a Suslin space and \mathcal{O} is its Borel σ -algebra, (2) is verified and (4) is also true for any probability measure λ .

Let $\bar{\mathcal{O}}$ be the σ -algebra of all *universally measurable* sets of Ω . i.e. $\bar{\mathcal{O}} = \bigcap_{\mu \in m^+(\Omega)} \hat{\mathcal{O}}$ where $m^+(\Omega)$ is the set of all probability measures on Ω .

We remark that our assumptions regarding Ω , \mathcal{O} and λ implies the existence and uniqueness of disintegrations of λ with respect to any σ -algebra contained in $\bar{\mathcal{O}}$.

If \mathcal{H} is any σ -algebra contained in $\bar{\mathcal{O}}$, let $(\lambda_w^{\mathcal{H}})_{w \in \Omega}$ denote a disintegration of λ with respect to \mathcal{H} .

Definition 97. Let \mathcal{S} and \mathcal{C} be two σ -algebras contained in $\bar{\mathcal{O}}$. We say \mathcal{C} is λ -stronger than \mathcal{S} if $\forall \lambda_w$, $\lambda_w^{\mathcal{C}}$ admits as disintegration with respect to \mathcal{C} , the family of measures $(\lambda_{w'}^{\mathcal{C}})_{w' \in \Omega}$.

The following proposition shows that the property “ \mathcal{C} is λ -stronger than \mathcal{S} ” does not depend on the chosen disintegrations.

Proposition 98. *Let \mathcal{S} and \mathcal{C} be two σ -algebras contained in $\bar{\mathcal{O}}$. Let $(\lambda_w^{\mathcal{C}})_{w \in \Omega}$ (resp. $(\lambda_w^{\mathcal{C}})_{w \in \Omega}$) and $(\chi_w^{\mathcal{C}})_{w \in \Omega}$ (resp. $(\chi_w^{\mathcal{C}})_{w \in \Omega}$) be two disintegrations of λ with respect to \mathcal{S} (resp. \mathcal{C}). If $\forall \lambda w, \lambda_w^{\mathcal{S}}$ has $(\lambda_{w'}^{\mathcal{C}})_{w' \in \Omega}$ for disintegration with respect to \mathcal{C} , then $\forall \lambda w, \chi_w^{\mathcal{S}}$ has $(\chi_{w'}^{\mathcal{C}})_{w' \in \Omega}$ for disintegration with respect to \mathcal{C} .*

Proof. Since $\forall \lambda w, \lambda_w^{\mathcal{S}}$ has $(\lambda_{w'}^{\mathcal{C}})_{w' \in \Omega}$ for disintegration with respect to \mathcal{C} , we have

$$(1) \quad \forall \lambda w, \lambda_w^{\mathcal{C}} = \int \lambda_{w'}^{\mathcal{C}} d\lambda_w^{\mathcal{S}}(dw')$$

$$(2) \quad \forall \lambda w, \text{ given } A \in \mathcal{C}, \forall_{\lambda_w^{\mathcal{S}}} w', \lambda_{w'}^{\mathcal{C}} \text{ is carried by } A \text{ or } \complement A \text{ according as } w' \in A \text{ or } w' \in \complement A.$$

To prove the proposition, we have only to prove that

$$(i) \quad \forall \lambda w, \chi_w^{\mathcal{S}} = \int \chi_{w'}^{\mathcal{C}} d\chi_w^{\mathcal{S}}(w')$$

$$(ii) \quad \forall \lambda w, \text{ given } A \in \mathcal{C}, \forall_{\chi_w^{\mathcal{S}}} w', \chi_{w'}^{\mathcal{C}} \text{ is carried by } A \text{ or } \complement A \text{ according as } w' \in A \text{ or } w' \in \complement A.$$

Because of the uniqueness of disintegrations, we have

$$(a) \quad \forall \lambda w, \lambda_w^{\mathcal{S}} = \chi_w^{\mathcal{S}}$$

$$(b) \quad \forall \lambda w, \lambda_w^{\mathcal{C}} = \chi_w^{\mathcal{C}} \text{ and therefore,}$$

$$(c) \quad \forall \lambda w, \forall_{\chi_w^{\mathcal{S}}} w', \lambda_{w'}^{\mathcal{C}} = \chi_{w'}^{\mathcal{C}}.$$

It is now easy to see that (i) and (ii) follow from (1) and (2) because of (a) and (c). □

Proposition 99. *Let \mathcal{C} be a σ -algebra contained in $\bar{\mathcal{O}}$. If \mathcal{C} is countably separating (in particular, if \mathcal{C} is countably generated), it is λ -stronger than all its sub σ -algebras.*

Proof. Let $\mathcal{S} \subset \mathcal{C}$ be a σ -algebra. Since \mathcal{O} is countably generated, we have

$$\forall \lambda w, \lambda_w^{\mathcal{S}} = \int \lambda_{w'}^{\mathcal{C}} d\lambda_w^{\mathcal{S}}(dw')$$

115 because of the transitivity of the conditional expectations. Therefore, since $w' \rightarrow \lambda_w^{\mathcal{C}}$ belongs to \mathcal{C} , to prove the proposition, it is sufficient to prove by proposition (3, § 7, 52) that $\forall \lambda w, \forall \lambda_w^{\mathcal{C}} w', \lambda_w^{\mathcal{C}}$ is carried by the \mathcal{C} -atom of w' . Since $(\lambda_w^{\mathcal{C}})_{w \in \Omega}$ is a disintegration of λ with respect to \mathcal{C} and since \mathcal{C} is countably separating, according to proposition (3, § 7, 53), $\forall \lambda w', \lambda_w^{\mathcal{C}}$ is carried by the \mathcal{C} -atom of w' . Therefore, $\forall \lambda w, \forall \lambda_w^{\mathcal{S}} w', \lambda_w^{\mathcal{C}}$ is carried by the \mathcal{C} -atom of w' . \square

Proposition 100. *Let \mathcal{C} and \mathcal{S} be two σ -algebras contained in $\bar{\mathcal{O}}$. If \mathcal{C} is λ -stronger than \mathcal{S} , it is λ -stronger than every sub σ -algebra of \mathcal{S} .*

Proof. Let \mathcal{S}' be a σ -algebra contained in \mathcal{S} . Since \mathcal{O} is countably generated, we have $\forall \lambda w, \lambda_w^{\mathcal{S}'} = \int \lambda_w^{\mathcal{S}} d\lambda_w^{\mathcal{S}'}(w')$ by the transitivity of the conditional expectations.

Since \mathcal{C} is λ -stronger than \mathcal{S} , $\forall \lambda w', \lambda_w^{\mathcal{S}}$ has $(\lambda_w^{\mathcal{C}})_{w'' \in \Omega}$ for disintegration with respect to \mathcal{C} . Therefore, $\forall \lambda w, \forall \lambda_w^{\mathcal{S}'} w', \lambda_w^{\mathcal{S}}$ has $(\lambda_w^{\mathcal{C}})_{w'' \in \Omega}$ for disintegration with respect to \mathcal{C} . Hence by proposition (7, § 1, 96), $\forall \lambda w, \lambda_w^{\mathcal{S}'}$ has $(\lambda_w^{\mathcal{C}})_{w'' \in \Omega}$ for disintegration with respect to \mathcal{C} .

This proves that \mathcal{C} is λ -stronger than \mathcal{S}' . \square

Definition 101. *A σ -algebra \mathcal{C} contained in $\bar{\mathcal{O}}$ is said to be λ -strong if it is λ -stronger than itself.*

We see by the above proposition, proposition (7, § 2, 100), that a λ -strong σ -algebra is λ -stronger than all its sub σ -algebras. From proposition (7, § 2, 99) we see that if a σ -algebra contained in $\bar{\mathcal{O}}$ is countably separating, it is μ -strong for any probability measure μ on \mathcal{O} , if Ω has the μ -compactness metrizable property.

Proposition 102. *Let \mathcal{C} be a σ -algebra contained in $\bar{\mathcal{O}}$. \mathcal{C} is λ -strong if and only if one of the two following equivalent conditions is true.* 116

- (i) $\forall \lambda w, \lambda_w^{\mathcal{C}}$ is disintegrated with respect to \mathcal{C} by the constant measure valued function $w' \rightarrow \lambda_w^{\mathcal{C}}$.
- (ii) $\forall \lambda w, \lambda_w^{\mathcal{C}}$ is ergodic on \mathcal{C} .

Proof. The equivalence of (i) and (ii) follows from proposition (7, § 1, 95).

Since \mathcal{O} is countably generated, by Corollary (3, § 7, 56), we have

$$\forall \lambda w, \forall \lambda_w^{\mathcal{C}} w', \lambda_{w'}^{\mathcal{C}} = \lambda_w^{\mathcal{C}}.$$

Let \mathcal{C} be λ -strong. Let us prove (i). $\forall \lambda w, \lambda_w^{\mathcal{C}}$ is disintegrated by $(\lambda_{w'}^{\mathcal{C}})_{w' \in \Omega}$ with respect to \mathcal{C} , and since $\forall \lambda w, \forall \lambda_w^{\mathcal{C}} w', \lambda_{w'}^{\mathcal{C}}, \lambda_w^{\mathcal{C}} = \lambda_w^{\mathcal{C}}$, we see that $\forall \lambda w, \lambda_w^{\mathcal{C}}$ is disintegrated by the constant measure valued function $w' \rightarrow \lambda_w^{\mathcal{C}}$, with respect to \mathcal{C} . This is (i)

Let us assume (i) and prove that \mathcal{C} is λ -strong.

$\forall \lambda w, \lambda_w^{\mathcal{C}}$ is disintegrated with respect to \mathcal{C} by the constant measure valued function $w' \rightarrow \lambda_w^{\mathcal{C}}$ and $\forall \lambda w, \forall \lambda_w^{\mathcal{C}} w', \lambda_{w'}^{\mathcal{C}} = \lambda_w^{\mathcal{C}}$.

Hence, $\forall \lambda w, \lambda_w^{\mathcal{C}}$ is disintegrated with respect to \mathcal{C} by the family $(\lambda_{w'}^{\mathcal{C}})_{w' \in \Omega}$. This proves that \mathcal{C} is λ -strong. \square

In the following proposition, we will be using the fact that if f is a λ -integrable function and if $(\mathcal{C}^n)_{n \in \mathbb{N}}$ is a decreasing sequence of σ -algebras contained in $\bar{\mathcal{O}}$ with $\mathcal{C} = \bigcap_{n=1}^{\infty} \mathcal{C}^n$, then $\forall \lambda w, f^{\mathcal{C}^n}(w) \rightarrow f^{\mathcal{C}}(w)$. This follows immediately from the convergence theorem for martingales with respect to a decreasing sequence of σ -algebras. See P.A. Meyer [1], Chap. V. T21.

Proposition 103. *Let $(\mathcal{C}^n)_{n \in \mathbb{N}}$ be a decreasing sequence of σ -algebras contained in $\bar{\mathcal{O}}$ and let $\mathcal{C} = \bigcap_{n=1}^{\infty} \mathcal{C}^n$. If $\forall n \in \mathbb{N}, \mathcal{C}^n$ is λ -strong then \mathcal{C} is also λ -strong.*

117 *Proof.* We have to prove that $\forall \lambda w, \lambda_w^{\mathcal{C}}$ is disintegrated with respect to \mathcal{C} by $(\lambda_{w'}^{\mathcal{C}})_{w' \in \Omega}$. We have only to check that

$$\forall \lambda w, \forall B \in \mathcal{O}, w' \rightarrow \int \chi_B(w'') \lambda_{w'}^{\mathcal{C}}(dw'')$$

is a conditional expectation of χ_B with respect to \mathcal{C} for the measure $\lambda_w^{\mathcal{C}}$. i.e. we have to only prove that $\forall \lambda w, \forall B \in \mathcal{O}$, for all $A \in \mathcal{C}$,

$$\int_A \chi_B(w') \lambda_w^{\mathcal{C}}(dw') = \int_A \left(\int \chi_B(w'') \lambda_{w'}^{\mathcal{C}}(dw'') \right) \lambda_w^{\mathcal{C}}(dw').$$

To prove this, it is sufficient to prove that

$$\forall B \in \mathcal{O}, \forall_{\lambda} w, \int_A \chi_B(w') \lambda_w^{\mathcal{C}}(dw') = \int_A \left(\int \chi_B(w'') \lambda_{w'}^{\mathcal{C}}(dw'') \right) \lambda_w^{\mathcal{C}}(dw')$$

for all $A \in \mathcal{C}$. For if we prove this, an application of Monotone class theorem will give the result, since \mathcal{O} is countably generated.

Let $B \in \mathcal{O}$. Since λ is a probability measure, χ_B is λ -integrable. Hence,

$$\forall_{\lambda} w' \cdot (\chi_B)^{\mathcal{C}^n}(w') \rightarrow (\chi_B)^{\mathcal{C}}(w').$$

i.e. $\forall_{\lambda} w', \int \chi_B(w'') \lambda_{w'}^{\mathcal{C}^n}(dw'') \rightarrow \int \chi_B(w'') \lambda_{w'}^{\mathcal{C}}(dw'')$. Hence,

$$\forall_{\lambda} w, \forall_{\lambda_{w'}^{\mathcal{C}^n}} w', \int \chi_B(w'') \lambda_{w'}^{\mathcal{C}^n}(dw'') \rightarrow \int \chi_B(w'') \lambda_{w'}^{\mathcal{C}}(dw'').$$

Therefore, by Lebesgue's dominated convergence theorem (which is applicable here since $\forall_{\lambda} w, \forall_{\lambda_{w'}^{\mathcal{C}^n}} w', \lambda_{w'}^{\mathcal{C}^n}$ is a probability measure for all n and hence $\int \chi_B(w'') \lambda_{w'}^{\mathcal{C}^n}(dw'') \leq 1$ and 1 is integrable with respect to $\lambda_{w'}^{\mathcal{C}}$ for λ -almost all w), we have $\forall_{\lambda} w$, for all $A \in \mathcal{C}$,

$$\int_A \left(\int \chi_B(w'') \lambda_{w'}^{\mathcal{C}^n}(dw'') \right) \lambda_w^{\mathcal{C}}(dw') \rightarrow \int_A \left(\int \chi_B(w'') \lambda_{w'}^{\mathcal{C}}(dw'') \right) \lambda_w^{\mathcal{C}}(dw').$$

Since $\forall n \in \mathbb{N}$, \mathcal{C}^n is λ -strong, and hence is λ -stronger than \mathcal{C} , $\forall_{\lambda} w$, $\lambda_w^{\mathcal{C}}$ is disintegrated with respect to \mathcal{C}^n by $(\lambda_{w'}^{\mathcal{C}})_{w' \in \Omega}$ for all $n \in \mathbb{N}$. Hence, $\forall_{\lambda} w, \forall B \in \mathcal{O}$, for all $A \in \mathcal{C}$, for all $n \in \mathbb{N}$,

$$\int_A \left(\int \chi_B(w'') \lambda_{w'}^{\mathcal{C}^n}(dw'') \right) \lambda_w^{\mathcal{C}}(dw') = \int_A \chi_B(w') \lambda_w^{\mathcal{C}}(dw').$$

Hence $\forall B \in \mathcal{O}, \forall_{\lambda} w$, for all $A \in \mathcal{C}$,

$$\int \chi_B(w') \lambda_w^{\mathcal{C}}(dw') = \int_A \left(\int \chi_B(w'') \lambda_{w'}^{\mathcal{C}}(dw'') \right) \lambda_w^{\mathcal{C}}(dw').$$

This is what we wanted to prove. \square

3 Choquet-type integral representations

In this section, we shall obtain a Choquet-type integral representation for probability measures which have a given family of probability measures as disintegration with respect to a given σ -algebra of $\bar{\mathcal{O}}$.

Definition 104. Let \mathcal{C} be a sub σ -algebra of $\bar{\mathcal{O}}$. \mathcal{C} is said to be universally strong if it is λ -strong for every probability measure λ on \mathcal{O} .

Let us fix a sub σ -algebra \mathcal{C} of $\bar{\mathcal{O}}$ which is universally strong, throughout this section. Let $(\lambda_w)_{w \in \Omega}$ be a family of probability measures on \mathcal{O} . Let us assume that the measure valued function $w \rightarrow \lambda_w$ belongs to \mathcal{C} , i.e. $\forall B \in \mathcal{C}, w \rightarrow \lambda_w(B) \in \mathcal{C}$.

$\forall w \in \Omega$, consider the set $\{w' \in \Omega \mid \lambda_{w'} = \lambda_w\}$. This set $\in \mathcal{C}$ and hence is a union of \mathcal{C} -atoms. Hence, we shall call it the molecule of w and write it as $\text{Mol } w$.

Let $\tilde{\Omega} = \{w \in \Omega \mid \lambda_w \text{ is disintegrated with respect to } \mathcal{C} \text{ by } (\lambda_{w'})_{w' \in \Omega} \text{ and is carried by } \text{Mol } w\}$.

119 Let $\mathcal{K} = \{\lambda \mid \lambda \text{ a probability measure on } \mathcal{O} \text{ such that } \lambda \text{ has } (\lambda_w)_{w \in \Omega} \text{ as disintegration with respect to } \mathcal{C}\}$.

Then \mathcal{K} is a convex set.

Proposition 105. If $\lambda \in \mathcal{K}$, λ is carried by $\tilde{\Omega}$ and $\lambda = \int_{\tilde{\Omega}} \lambda_w d\lambda(w)$. If μ is any probability measure carried by $\tilde{\Omega}$ and if $\rho = \int_{\tilde{\Omega}} \lambda_w d\mu(w)$, then $\rho \in \mathcal{K}$.

Proof. Let $\lambda \in \mathcal{K}$. Since \mathcal{C} is universally strong, it is in particular λ -strong λ has $(\lambda_w)_{w \in \Omega}$ for disintegration with respect to \mathcal{C} . Hence, $\forall \lambda w, \lambda_w$ is disintegrated with respect to \mathcal{C} by $(\lambda_{w'})_{w' \in \Omega}$. Moreover, by Corollary (3, § 7, 56),

$$\forall \lambda w, \forall \lambda_w w', \lambda_{w'} = \lambda_w.$$

This shows that λ is carried by $\tilde{\Omega}$. Since $\lambda = \int_{\tilde{\Omega}} \lambda_w \lambda(dw)$ and since λ is carried by $\tilde{\Omega}$, $\lambda = \int_{\tilde{\Omega}} \lambda_w \lambda(dw)$. The rest of the proposition follows immediately from proposition (7, § 1, 96). \square

Theorem 106. *The extreme points of \mathcal{K} are precisely the measures λ_w where $w \in \tilde{\Omega}$.*

Proof. Let us first show that $\forall w \in \tilde{\Omega}$, λ_w is extreme.

Let $w \in \tilde{\Omega}$. Then $\lambda_w \in \mathcal{K}$. Let μ and ν be two measures $\in \mathcal{K}$ and t be a real number with $0 < t < 1$ such that

$$\lambda_w = t\mu + (1-t)\nu.$$

$\text{Mol } w \in \mathcal{C}$ and $\lambda_w(\text{Mol } w) = 1$ since λ_w is carried by $\text{Mol } w$. Hence μ and ν are also carried by $\text{Mol } w$. By the previous proposition 7, § 3, 105), $\mu = \int \lambda_{w'}\mu(dw')$ and $\nu = \int \lambda_{w'}\nu(dw')$. Since μ and ν are carried by $\text{Mol } w$, the integration is actually over only $\text{Mol } w$. Thus,

$$\mu = \int_{\text{Mol } w} \lambda_{w'}\mu(dw') \text{ and } \nu = \int_{\text{Mol } w} \lambda_{w'}\nu(dw').$$

But for $w' \in \text{Mol } w$, $\lambda_{w'} = \lambda_w$ and hence $\mu = \lambda_w = \nu$.

Thus λ_w is extreme $\forall w \in \tilde{\Omega}$.

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Now, let λ be an extreme point of \mathcal{K} . Let us show that there exists a $w \in \tilde{\Omega}$ such that $\lambda_w = \lambda$.

First of all, we claim that λ must be ergodic on \mathcal{C} . For, if not, $\exists A \in \mathcal{C}$ such that $0 < \lambda(A) < 1$,

$$\lambda = \lambda(A) \cdot \frac{\chi_A \cdot \lambda}{\lambda(A)} + \lambda(\complement A) \cdot \frac{\chi_{\complement A} \cdot \lambda}{\lambda(\complement A)}.$$

Thus, λ is a convex combination of the measures $\frac{\chi_A \cdot \lambda}{\lambda(A)}$ and $\frac{\chi_{\complement A} \cdot \lambda}{\lambda(\complement A)}$.

Since $A \in \mathcal{C}$, and $\lambda \in \mathcal{K}$, $\frac{\chi_A \cdot \lambda}{\lambda(A)}$ also $\in \mathcal{K}$. Similarly, $\frac{\chi_{\complement A} \cdot \lambda}{\lambda(\complement A)}$ also

belongs to \mathcal{K} . These two measures are not the same since $\frac{\chi_A \cdot \lambda}{\lambda(A)}$ is

carried by A and $\frac{\chi_{\complement A} \cdot \lambda}{\lambda(\complement A)}$ is carried by $\complement A$. Thus, we get a contradiction

to the fact that λ is extreme. Hence λ must be ergodic on \mathcal{C} . Hence by proposition (7, § 1, 95) λ is disintegrated with respect to \mathcal{C} by the

constant measure valued function $w \rightarrow \lambda_w^{\mathcal{C}} = \lambda$. It follows therefore, by uniqueness of disintegrations that $\forall \lambda w, \lambda_w = \lambda$. By the previous proposition λ is carried by $\tilde{\Omega}$. Hence there exists a $w \in \tilde{\Omega}$ such that $\lambda_w = \lambda$. \square

In view of theorem (7, § 3, 106) and proposition (7, § 3, 105) we see that the integral representation of $\lambda \in \mathcal{K}$ as $\int_{\tilde{\Omega}} \lambda_w \lambda(dw)$ is indeed of the same type as the one considered by Choquet. But, we have not deduced our result from Choquet's theory.

Let us now prove a kind of uniqueness theorem.

Let $\tilde{\Omega}'$ be the quotient set of $\tilde{\Omega}$ by the molecules. Let p be the canonical mapping from $\tilde{\Omega}$ to $\tilde{\Omega}'$. Let \mathcal{C}° be the σ -algebra on $\tilde{\Omega}'$ consisting of sets whose inverse image under p belongs to \mathcal{C} . If $w \in \tilde{\Omega}$, let us denote by \dot{w} the element $p(w)$ of $\tilde{\Omega}'$. $\forall \dot{w} \in \tilde{\Omega}'$ define the measure $\lambda_{\dot{w}}$ on \mathcal{C}° as the image measure of λ_w under the mapping p where w is such that $p(w) = \dot{w}$.

121 This is independent of the choice of w in $p^{-1}(\dot{w})$, for if $p(w_1) = p(w_2)$, then $\lambda_{w_1} = \lambda_{w_2}$. If μ is any measure on \mathcal{C} , let us denote by $\dot{\mu}$ its image measure under p on \mathcal{C}° . We note that if $A \in \mathcal{C}^\circ$, $\lambda_{\dot{w}}(A) = 1$ or 0 according as $\dot{w} \in A$ or not. For, if $\dot{w} \in A$, then $w \in p^{-1}(A)$ where w is such that $p(w) = \dot{w}$. Hence $\text{Mol } w \subset p^{-1}(A)$. Therefore, $\lambda_{\dot{w}}(A) = \lambda_w(p^{-1}(A)) = 1$ since λ_w is carried by $\text{Mol } w$. If $\dot{w} \notin A$, $\dot{w} \in \complement A$ and by the same argument $\lambda_{\dot{w}}(\complement A) = 1$. Hence $\lambda_{\dot{w}}(A) = 0$.

Theorem 107. *Let $\lambda \in \mathcal{K}$. If μ is any probability measure carried by $\tilde{\Omega}$ such that $\lambda = \int \lambda_w \mu(dw)$, then $\dot{\mu} = \dot{\lambda}$ on \mathcal{C}° .*

Proof. Let $\lambda = \int_{\tilde{\Omega}} \lambda_w \mu(dw)$. We first see that $\dot{\lambda} = \int_{\tilde{\Omega}'} \lambda_{\dot{w}} \dot{\mu}(d\dot{w})$. For let $A \in \mathcal{C}^\circ$

$$\begin{aligned} \dot{\lambda}(A) &= \lambda(p^{-1}(A)) = \int \lambda_w(p^{-1}(A)) \mu(dw) \\ &= \int \lambda_{\dot{w}}(A) \mu(dw) \\ &= \int \lambda_{p(w)}(A) \mu(dw) \end{aligned}$$

$$= \int \lambda_{\dot{w}}(A) \dot{\mu}(d\dot{w}).$$

Hence,

$$\lambda = \int_{\tilde{\Omega}} \lambda_{\dot{w}} \dot{\mu}(d\dot{w}).$$

For any $A \in \mathcal{C}^\circ$, $\lambda(A) = \int_{\tilde{\Omega}} \lambda_{\dot{w}}(A) \dot{\mu}(d\dot{w})$. Since $\lambda_{\dot{w}}(A) = 1$ or 0 according as $\dot{w} \in A$ or $\complement A$, the integration is actually only over A ; i.e.

$$\lambda(A) = \int_A \lambda_{\dot{w}}(A) \dot{\mu}(d\dot{w}).$$

But, $\int_A \lambda_{\dot{w}}(A) \dot{\mu}(d\dot{w}) = \dot{\mu}(A)$ since $\lambda_{\dot{w}}(A) = 1$ for $\dot{w} \in A$. Hence,

$$\lambda(A) = \dot{\mu}(A), \quad \forall A \in \mathcal{C}^\circ$$

and therefore, $\lambda = \dot{\mu}$. □

4 The Usual σ -Algebras

In this section, we shall define the σ -algebras that occur in the theory of Brownian motion on the real line and prove some properties of them. 122

Let $\Omega_{[0,+\infty)}$ be the set of all real valued continuous functions on $[0, +\infty)$. Let D be a countable dense subset of $[0, +\infty)$. For $t \in [0, +\infty)$ define the mapping π_t from $\Omega_{[0,+\infty)}$ to \mathbb{R} as $\pi_t(w) = w(t)$, where w is an element of $\Omega_{[0,+\infty)}$. π_t is called the t 'th projection. Let \mathfrak{T}_D be the topology on $\Omega_{[0,+\infty)}$ which is the coarsest making all the $(\pi_t)_{t \in D}$ continuous. Let \mathcal{O}_D be the Borel σ -algebra of $\Omega_{[0,+\infty)}$ for the topology \mathfrak{T}_D . It can be easily checked that \mathcal{O}_D is the smallest σ -algebra making the projections $(\pi_t)_{t \in D}$ measurable. Since $\forall t \in [0, +\infty)$, $\pi_t = \lim_{\substack{t_n \rightarrow t \\ t_n \in D}} \pi_{t_n}$, it follows immediately that \mathcal{O}_D is also the smallest σ -algebra making all the projections $(\pi_t)_{t \in [0,+\infty)}$ measurable.

Let \mathfrak{T}_P be the topology of pointwise convergence on $\Omega_{[0,+\infty)}$ i.e. \mathfrak{T}_P is the coarsest topology making all the $(\pi_t)_{t \in [0,+\infty)}$ continuous. Let \mathcal{O}_P be

the Borel σ -algebra of $\Omega_{[0,+\infty)}$ in this topology \mathfrak{S}_p . Let \mathfrak{S} be the topology of uniform convergence on compact sets of $[0, +\infty)$. Let \mathcal{O} be the Borel σ -algebra of $\Omega_{[0,+\infty)}$ in this topology \mathfrak{S} . We easily see that \mathfrak{S}_D is coarser than \mathfrak{S}_p which in turn is coarser than \mathfrak{S} . Hence $\mathcal{O}_D \subset \mathcal{O}_p \subset \mathcal{O}$.

Now, $\Omega_{[0,+\infty)}$ is a separable Fréchet space under the topology \mathfrak{S} and hence is a Polish space i.e. is homeomorphic to a complete separable metric space. It is a theorem that the Borel σ -algebra of a Polish space is the same as the Borel σ -algebra of any coarser Hausdorff topology. Hence $\mathcal{O}_D = \mathcal{O}_p = \mathcal{O}$.

Thus the smallest σ -algebra making all the $(\pi_t)_{t \in [0,+\infty)}$ measurable coincides with the Borel σ -algebra of the topology of pointwise convergence on $[0, +\infty)$ and it is countably generated since \mathcal{O} is countably generated.

123 Let \mathcal{U}' be the σ -algebra on $\Omega_{[0,+\infty)}$ generated by $(\pi_s)_{s \leq t}$ where $t \in [0, +\infty)$. Let $\Omega_{[0,t]}$ be the space of all real valued continuous functions on $[0, t]$. For $0 \leq s \leq t$, let π'_s be the map defined on $\Omega_{[0,t]}$ as $\pi'_s(w) = w(s)$ where $w \in \Omega_{[0,t]}$. As above, we can see that the Borel σ -algebra of $\Omega_{[0,t]}$ for the topology of pointwise convergence on $[0, t]$ is the same as the σ -algebra generated by $(\pi'_s)_{0 \leq s \leq t}$. Let us denote this σ -algebra by \mathcal{O}^t . As above we can see that \mathcal{O}^t is countably generated.

Let $p : \Omega_{[0,+\infty)}$ \rightarrow $\Omega_{[0,t]}$ be the restriction map. It is easily checked that \mathcal{U}' is equal to $p^{-1}(\mathcal{O}^t)$ where $p^{-1}(\mathcal{O}^t)$ is the σ -algebra consisting of sets $p^{-1}(A)$ as A varies over \mathcal{O}^t . Since \mathcal{O}^t is countably generated, \mathcal{U}' is also countably generated.

Let $w \in \Omega_{[0,+\infty)}$. The \mathcal{U}' -atom of w is $p^{-1}(p(w))$ i.e. the set of all trajectories which coincide with w upto time t .

Proposition 108. $A \in \mathcal{U}' \iff A \in \mathcal{O}$ and is a union of \mathcal{U}' -atoms.

Proof. Let $A \in \mathcal{U}'$. Then clearly $A \in \mathcal{O}$ and is a union of \mathcal{U}' -atoms. We have to prove only the other way.

Let $(f_n)_{n \in \mathbb{N}}$ be a countable number of functions generating the σ -algebra \mathcal{U}' . Consider the mapping $\varphi : \Omega_{[0,+\infty)}$ \rightarrow $\mathbb{R}^{\mathbb{N}}$ given by $\varphi(w) = (f_n(w))_{n \in \mathbb{N}}$. Since $\forall n \in \mathbb{N}$, $f_n \in \mathcal{U}'$, φ also $\in \mathcal{U}'$ and hence is Borel measurable i.e. measurable with respect to \mathcal{O} . Hence $\varphi(\Omega_{[0,+\infty)})$ is a Suslin subset of $\mathbb{R}^{\mathbb{N}}$ since $\Omega_{[0,+\infty)}$ is a Polish space for the topology \mathfrak{S}

and \mathcal{O} is its Borel σ -algebra. Since $A \in \mathcal{O}$, A is Suslin and hence $\varphi(A)$ is Suslin. Similarly $\varphi(\mathbb{C}A)$ is also Suslin. Since A is union of \mathcal{U}^t -atoms, it is easily checked that $\varphi(\Omega_{[0,+\infty)})\varphi(A)$ is precisely $\varphi(\mathbb{C}A)$. Thus the Suslin subsets $\varphi(A)$ and $\varphi(\mathbb{C}A)$ are complementary in $\varphi(\Omega_{[0,+\infty)})$ and hence are Borel. Since A is a union of \mathcal{U}^t -atoms, $A = \varphi^{-1}(\varphi(A))$. Since $\varphi \in \mathcal{U}^t$ and $\varphi(A)$ is Borel, it follows that $\varphi^{-1}(\varphi(A))$ i.e. $A \in \mathcal{U}^t$. \square 124

Let $\forall t \in [0, +\infty)$, \mathcal{C}^t be the σ -algebra $\bigcap_{\epsilon > 0} \mathcal{U}^{t+\epsilon}$.

Corollary 109. $A \in \mathcal{C}^t \implies A \in \mathcal{O}$ and is a union of \mathcal{C}^t -atoms.

Proof. If $A \in \mathcal{C}^t$, then $A \in \mathcal{O}$ and is a union of \mathcal{C}^t -atoms. Conversely if $A \in \mathcal{O}$ and is a union of \mathcal{C}^t -atoms, it is $\forall \epsilon > 0$, a union of $\mathcal{U}^{t+\epsilon}$ -atoms and hence by the above proposition, $A \in \mathcal{U}^{t+\epsilon} \forall \epsilon > 0$. Hence $A \in \mathcal{C}^t$. \square

Proposition 110. The \mathcal{C}^t -atom of w consists of precisely the trajectories which coincide with w a little beyond t . i.e.

$$\mathcal{C}^t - \text{atom of } w = \{w' \mid \exists t_{w'} > t \text{ such that } w' = w \text{ in } [0, t_{w'}]\}.$$

Proof. Let $B = \{w' \mid \exists t_{w'} > t \text{ such that } w' = w \text{ in } [0, t_{w'}]\}$

$$\begin{aligned} B &= \bigcup_{\epsilon > 0} \{w' \mid w' = w \text{ in } [0, t + \epsilon]\} \\ &= \bigcup_{\epsilon > 0} \{\mathcal{U}^{t+\epsilon} - \text{atom of } w\}. \end{aligned}$$

Let A be any set $\in \mathcal{C}^t$ containing w . Then, $\forall \epsilon > 0$, $A \in \mathcal{U}^{t+\epsilon}$ and hence A contains the $\mathcal{U}^{t+\epsilon}$ -atom of w , $\forall \epsilon > 0$. Hence $A \supset B$.

Therefore, to prove the proposition, sufficient to prove that $B \in \mathcal{C}^t$.

$$\begin{aligned} B &= \bigcup_{\epsilon > 0} \{\mathcal{U}^{t+\epsilon} - \text{atom of } w\} \\ &= \bigcup_{n \in \mathbb{N}} \{\mathcal{U}^{t+\frac{1}{n}} - \text{atom of } w\} \\ &= \bigcup_{m \in \mathbb{N}} \bigcup_{\substack{\frac{1}{n} < \frac{1}{m} \\ n \in \mathbb{N}}} \{\mathcal{U}^{t+\frac{1}{n}} - \text{atom of } w\} \end{aligned}$$

$\forall m \in \mathbb{N}, \bigcup_{\substack{\frac{1}{n} < \frac{1}{m} \\ n \in \mathbb{N}}} \{\mathcal{U}^{t+\frac{1}{n}} - \text{atom of } w\} \in \mathcal{U}^{t+\frac{1}{m}}$ and hence,

$$\bigcap_{m \in \mathbb{N}} \bigcup_{\substack{\frac{1}{n} < \frac{1}{m} \\ n \in \mathbb{N}}} \{\mathcal{U}^{t+\frac{1}{n}} - \text{atom of } w\} \in \mathcal{C}^t.$$

□

125 One can prove that $\forall t, \mathcal{C}^t$ is not countably separating. For a proof see L. Schwartz [1], page 161.

However, $\forall t, \mathcal{C}^t$ is universally strong. For, since $\forall \epsilon > 0, \mathcal{U}^{t+\epsilon}$ is countably generated, it is universally strong by proposition (7, § 2, 99). Since $\mathcal{C}^t = \bigcap_{n \in \mathbb{N}} \mathcal{U}^{t+\frac{1}{n}}$, by proposition (7, § 2, 103), it follows that \mathcal{C}^t is universally strong $\forall t \in [0, +\infty)$.

5 Further results on regular disintegrations

In this section, let us assume that Ω is a compact metrizable space and \mathcal{O} is its Borel σ -algebra. Let $(\mathcal{C}^t)_{t \in \mathbb{R}}$ be a right continuous increasing family of σ -algebras contained in \mathcal{O} which is the σ -algebra of all universally measurable sets of Ω . Let λ be a probability measure on \mathcal{O} .

Note that a unique regular disintegration of λ with respect to $(\mathcal{C}^t)_{t \in \mathbb{R}}$ exists.

Let us assume that $\forall t \in \mathbb{R}, \mathcal{C}^t$ is λ -strong.

Theorem 111. *Let $(\lambda_w^t)_{\substack{w \in \Omega \\ t \in \mathbb{R}}}$ be a regular disintegration of λ with respect to $(\mathcal{C}^t)_{t \in \mathbb{R}}$. Then $\forall \lambda w, \forall s, \lambda_w^s$ has a regular disintegration with respect to $(\mathcal{C}^t)_{t \in \mathbb{R}}$ given by*

$$\begin{aligned} (t, w') &\rightarrow \lambda_{w'}^t \text{ for } t \geq s \\ (t, w') &\rightarrow \lambda_w^s \text{ for } t < s. \end{aligned}$$

Proof. Let t be fixed. Since \mathcal{C}^t is λ -strong, $\forall \lambda w, \lambda_w^t$ has $(\lambda_w^t)_{w' \in \Omega}$ for disintegration with respect to \mathcal{C}^t . Since a set of λ -measure zero is also $\forall \lambda w, \forall s$, of λ_w^s -measure zero, it follows therefore that

$\forall \lambda w, \forall s, \forall \lambda_w^s w', \lambda_{w'}^t$ has $(\lambda_{w''}^t)_{w'' \in \Omega}$ for disintegration with respect to \mathcal{C}^t .

By theorem (6, § 2, 94), we have

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$$\forall \lambda w, \forall s \leq t, \lambda_w^s = \int \lambda_{w'}^t \lambda_w^s(dw').$$

Hence, by proposition (7, § 1, 96), $\forall \lambda w, \forall s \leq t, \lambda_w^s$ has $(\lambda_{w''}^t)_{w'' \in \Omega}$ for disintegration with respect to \mathcal{C}^t . Thus, $\forall t, \forall \lambda w, \forall s \leq t, \lambda_w^s$ has $(\lambda_{w''}^t)_{w'' \in \Omega}$ for disintegration with respect to \mathcal{C}^t .

Therefore, $\forall \lambda w, \forall t$, a dyadic rational $\forall s \leq t, \lambda_w^s$ has $(\lambda_{w''}^s)_{w'' \in \Omega}$ for disintegration with respect to \mathcal{C}^t .

Now, let μ be a probability measure on \mathcal{O} and let $(\delta_w^t)_{w \in \Omega}$ be a disintegration of μ with respect to $\mathcal{C}^t \forall$ dyadic t .

For any $t \in \mathbb{R}$, define

$$\delta_w^t = \begin{cases} \text{vague } \lim \delta_w^{\tau_n(t)}, & \text{if it exists} \\ 0, & \text{if not} \end{cases}$$

Then, since Ω is compact metrizable, we can easily check that $(\delta_w^t)_{w \in \Omega}$ is a regular disintegration of μ with respect to $(\mathcal{C}^t)_{t \in \mathbb{R}}$.

Now we have,

$$\forall \lambda w, \forall t \in \mathbb{R}, \lambda_w^t = \text{vague } \lim_{n \rightarrow \infty} \lambda_w^{\tau_n(t)}.$$

Hence, since $\forall \lambda w, \forall s, \lambda_w^s$ has a disintegration $(\lambda_{w''}^t)_{w'' \in \Omega}$ with respect to \mathcal{C}^t for all t dyadic $\geq s$, by the preceding paragraph, $\forall \lambda w, \forall s, \lambda_w^s$ has $(\lambda_{w''}^t)_{w'' \in \Omega}$ for a regular disintegration with respect to \mathcal{C}^t for all $t \geq s$.

By the theorem of trajectories, i.e. by theorem (6, § 2, 93),

$$\forall \lambda w, \forall s, \forall \lambda_w^s w', \forall t \leq s, \lambda_{w'}^t = \lambda_w^t.$$

In particular,

$$\forall \lambda w, \forall s, \forall \lambda_w^s w', \lambda_{w'}^s = \lambda_w^s.$$

Since \mathcal{C}^s is λ -strong. $\forall \lambda w, \lambda_w^s$ is disintegrated by $(\lambda_{w''}^s)_{w'' \in \Omega}$ with respect to \mathcal{C}^s . Since $\forall \lambda w, \forall s, \forall \lambda_w^s w', \lambda_{w'}^s = \lambda_w^s, \forall \lambda w, \forall \lambda w, \forall s, \lambda_w^s$ is

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disintegrated with respect to \mathcal{C}^s by the constant measure valued function $w' \rightarrow \lambda_w^s$. Hence, by proposition (7, § 1, 95), $\forall \lambda w, \forall s, \lambda_w^s$ is ergodic on \mathcal{C}^s . Hence $\forall \lambda w, \forall s, \forall t \leq s, \lambda_w^s$ is ergodic on \mathcal{C} . Hence, again by the proposition (7, § 1, 95), $\forall \lambda w, \forall s, \forall t \leq s, \lambda_w^s$ is disintegrated with respect to \mathcal{C}^t by the constant measure valued function $w' \rightarrow \lambda_w^s$.

Thus, we have proved that $\forall \lambda w, \forall s, \lambda_w^s$ has with respect to $(\mathcal{C}^t)_{t \in \mathbb{R}}$ a disintegration given by $(t, w') \rightarrow \lambda_{w'}^t$ for $t \geq s$ and $(t, w') \rightarrow \lambda_w^s$ for $t < s$.

We have to check only that this is a regular disintegration. Since $(t, w) \rightarrow \lambda_w^t$ is a regular disintegration of λ , we have to check only the right continuity at the point s . i.e., we have to only prove that

$$\forall B \in \mathcal{O}, \forall \lambda w, \forall s, \forall \lambda_w^s w', \lim_{t \downarrow s} \lambda_{w'}^t(B) = \lambda_w^s(B).$$

But this follows immediately from the fact that $\forall \lambda w', \lim_{t \downarrow s} \lambda_{w'}^t(B) = \lambda_{w'}^s(B)$ and

$$\forall \lambda w, \forall s, \forall \lambda_w^s w', \lambda_{w'}^s = \lambda_w^s.$$

□

Corollary 112. *Let f be a regular supermartingale adapted to $(\mathcal{C}^t)_{t \in \mathbb{R}}$. Then $\forall \lambda w, \forall s, f$ remains a supermartingale on the set $\Omega \times [s, +\infty)$, for the measure λ_w^s .*

Proof. From the above theorem, we have $\forall \lambda w, \forall s, \forall$ pair $\{t, t'\}$ with $s \leq t \leq t', \forall \lambda_w^s w'$.

$$E^{\lambda_w^s}(f^{t'} | \mathcal{C}^t)(w') = \lambda_{w'}^t(f^{t'})$$

where $E^{\lambda_w^s}(f^{t'} | \mathcal{C}^t)(w')$ stands for the value at w' of the conditional expectation of $f^{t'}$ with respect to \mathcal{C}^t for the measure λ_w^s .

Because f is a supermartingale, $\forall \lambda w, \lambda_w^t(f^{t'}) \leq f^t(w)$. Hence

$$\forall \lambda w, \forall s, \forall \lambda_w^s w' \cdot \lambda_{w'}^t(f^{t'}) \leq f^t(w').$$

Hence $\forall \lambda w, \forall s, \forall$ pair $\{t, t'\}$ with $s \leq t \leq t', \forall \lambda_w^s w', E^{\lambda_w^s}(f^{t'} | \mathcal{C}^t)(w') \leq f^t(w')$.

This shows that $\forall \lambda w, \forall s, f$ is a supermartingale on $\Omega \times [s, +\infty)$ for the measure λ_w^s .

The regularity of f with respect to λ_w^s follows from that of f with respect to λ . \square

Remark 113. Under the assumptions that \mathcal{C}^t is λ -strong $\forall t \in \mathbb{R}$ and Ω , compact metrizable, note that the above corollary is stronger than theorem (6, § 2, 91).

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