STRAIN GAUGE MEASUREMENTS

INTRODUCTION

Strain gauge measurement is a point strain measurement method developed in the USA c.a. 1938 independently by E. Simmons and A. Ruge. It enables determination of strain value in certain point and also other quantities related to relative strain such as stresses and internal forces. Each kind of devices used in those measurements (**strain gauges**) has its own mechanism of measurement, its gauge basis and precision in Hooke's Law validity range. Strain gauges are widely used in machine construction, civil engineering, medicine etc.

The clue part of the strain gauge is a **sensor fastened to the surface** of examined body in such way so that deformation of the sensor and deformation of the body are identical. This deformation (strain) can be determined in a **mechanical** or **electrical** way.

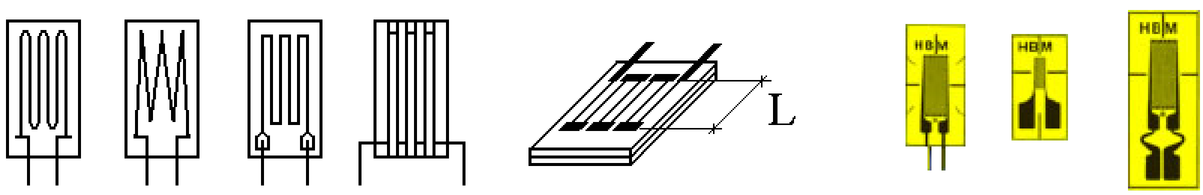
One of the most important property of a strain gauge is its **basis** (gauge length). It is an initial length L_0 to which its increment ΔL_0 is related. Strain calculated using simple relation $\varepsilon = \Delta L_0 / L_0$ is only an estimation of true strain – it is an **average strain along the gauge length**. This is why in case of stress concentration small basis is used (0,5-3 mm). In case of linear distribution of stresses larger gauge lengths are used (5-30 mm). Relatively large basis (over 30 mm) is used in case of determining mechanical properties of a body which takes place usually when stress distribution in the body is uniform.

There are several general types of strain gauges depending on their construction and physical phenomenon used in strain determination:

- **mechanical** strain gauges
- **mechanical-optical** strain gauges
- **induction** strain gauges
- **capacitance** strain gauges

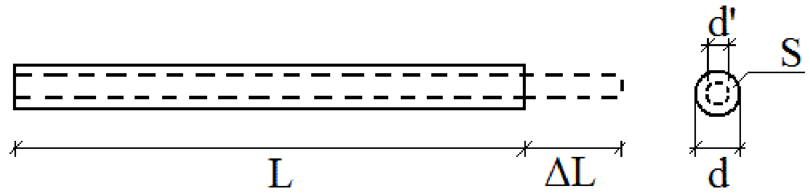
STRESS – RESISTANCE STRAIN GAUGES

In stress state analysis of elements of machines glued **stress-resistance** strain gauges are commonly used. Those gauges are cheap, universal, very sensitive and precise, they exhibit no mechanical inertia and can be glued practically in every place on the machine. Mechanism of strain determination is very simple – strain gauge is in fact a (relatively) long electrical conductor. Its resistance depends mainly on its length – **any change of length** (caused by deformation of a surface to which strain gauge is attached) **causes change of resistance** which can be easily measured. The conductor is made of very thin (0,02 – 0,04 mm diameter) wire made of certain alloy (i.e. constantan 60% Cu, 40% Ni). Both sides of the wire are covered with a foil. Conductor has always a form of loops or grid as shown below:



Grid shaped strain gauges are insensitive on transverse deformation which is its advantage over loop shaped gauges. Each sensor ends with a copper ending of much larger cross-section and very low resistance – they allow connecting (i.e. soldering) the sensors with proper gauges. Few types of endings proposed by HBM company are visible above.

Let's analyze deformation and resistance change of a round wire:



L – initial conductor's length
ΔL – conductor's length increment
d – initial conductor's diameter
d' – diameter after deformation $d' = d(1-\nu)$, ν – Poisson's coefficient
S – initial conductor's cross-section area
ρ – specific electrical resistance (resistivity)

Resistance is given by a formula shown below. After calculating logarithm of both sides of equation and after differentiating it and making some substitutions:

$$R = \rho \frac{L}{S} \qquad / \ln(\cdot)$$

$$\ln R = \ln\left(\rho \frac{L}{S}\right) = \ln \rho + \ln L - \ln S \qquad / \frac{\mathrm{d}}{\mathrm{d}x} \quad , \quad x = R, \rho, L, S$$

$$\frac{\mathrm{d}R}{R} = \frac{\mathrm{d}\rho}{\rho} + \frac{\mathrm{d}L}{L} - \frac{\mathrm{d}S}{S}$$

$$S = \frac{\pi d^2}{4} \quad \Rightarrow \quad \mathrm{d}S = \frac{\pi 2d}{4}\mathrm{d}d = \frac{\pi d}{2}\mathrm{d}d \quad \Rightarrow \quad \frac{\mathrm{d}S}{S} = 2\frac{\mathrm{d}d}{d}$$

$$\frac{\mathrm{d}R}{R} = \frac{\mathrm{d}\rho}{\rho} + \frac{\mathrm{d}L}{L} - 2\frac{\mathrm{d}d}{d}$$

we can substitute infinitesimal increments dx with finite increments Δx:

$$\frac{\Delta R}{R} = \frac{\Delta \rho}{\rho} + \frac{\Delta L}{L} - 2\frac{\Delta d}{d}$$

Transverse strain can be expressed by longitudinal strain using Hooke's Law: $\frac{\Delta d}{d} = -\nu \frac{\Delta L}{L}$

Thus we obtain:

$$\frac{\Delta R}{R} = \frac{\Delta L}{L} \underbrace{\left[1 + 2\nu + \frac{\frac{\Delta\rho}{\rho}}{\frac{\Delta L}{L}}\right]}_{k=\text{const.}} \quad \Rightarrow \quad \frac{\Delta R}{R} = k\frac{\Delta L}{L} \approx k\varepsilon$$

So relative deformation (which is an estimation of average strain) is proportional to relative resistance change and can be calculated in an easy way. Parameter k is constant and depends only on material the conductor is made of. It is called **strain gauge sensitivity coefficient** (usually k = 1,6 – 3,6). The most important condition that must be fulfilled so the strain gauge worked properly is identical deformation of the sample, glue, paper or foil layers and the conductor itself. This is why a proper glue must be used. It has to be:

- waterproof
- high temperature insensitive
- it should not exhibit creeping under long-term loading
- it should exhibit no hysteresis (stable behavior under cyclic loading)

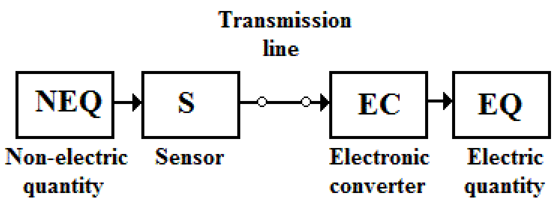
Basic properties of two exemplar glues (offered by Hottinger Baldwin Messtechnik Darmstads) are listed below:

Glue name	Application	Basic component	Temperature range for practical usage [°C]	Setting time in 20°C [s]	Young Mudulus [kN/mm^2]
X60	Experimental stress state analysis	metacryl	-55÷65	30	4,5÷6
Z70	Experimental stress state analysis	cyanoacryl	-55÷100	15	3

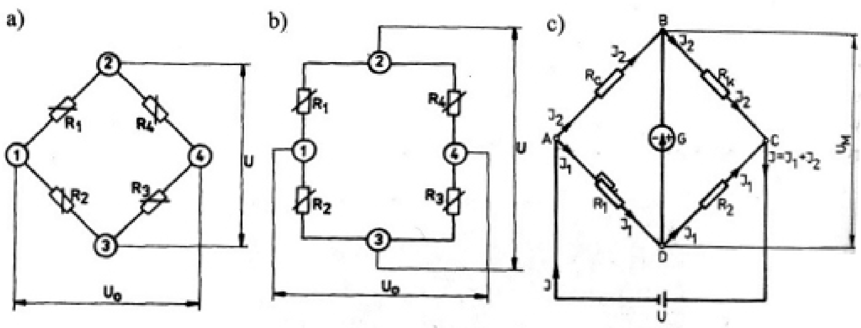
Place on the examined sample where strain gauge is to be fastened should be prepared properly. First of all it should be cleaned (especially fat should be removed). Then a short layer of glue should be placed and the sensor (which should be already glued to a foil) should be put on it. After that it should be covered by an waterproof material – i.e. beeswax or special kind of resin.

WHEATSTONE BRIDGE CIRCUIT AND ITS APLICATION IN STRAIN GAUGE MEASUREMENTS

General scheme of stress-resistance strain gauge is show below:



In spite of fact that there is a large variety of available types of strain gauges (different construction, different applications) they all base on common mechanism of work and use common construction elements. Fundamental element of each device is so called **Wheatstone bridge circuit** powered by alternating or direct electric current allowing resistance increment measurement under static or dynamical loading. Some examples of such bridge circuits are shown below:



In case c) two branches of the bridge circuit are made of two strain gauges (R_c – active, R_k – compensating, see below) and two other internal branches have resistances R_1 and R_2. Galvanometer is installed in BD branch. Values of resistances can be chosen in such way that no current appears in BD – since there is no current in BD, potential difference (voltage) in B and D equals 0. Initial current J caused by external voltage U source is divided into J_1 flowing from A through D to C, and

J_2 flowing from A through B to C. Since potential in B and in D are equal and also potential in A and C are equal thus potential fall along AB is the same as along AD and also falls along BC is equal as along DC. We can write:

$$\begin{cases} \Delta U_{AB} = J_2 R_c = J_1 R_1 = \Delta U_{AD} \\ \Delta U_{BC} = J_2 R_k = J_1 R_2 = \Delta U_{DC} \end{cases} \Rightarrow \quad \frac{R_c}{R_k} = \frac{R_1}{R_2}$$

We can see that circuit is in an equilibrium state (no current in BD, galvanometer indicates zero) when ratio of resistances of active and compensating strain gauges are equal ratio of resistances of the other two branches of the circuit.

Any deformation ε of a body on which strain gauge is glued cause change of resistance ΔR_c :

$$\varepsilon = \frac{\Delta R_c}{R_c} \frac{1}{k} \qquad R_c' = R_c + \Delta R_c \quad -\text{strain gauge resistance after deformation}$$

When it is possible to determine the value of R_c and ΔR_c directly from screen of galvanometer (scaled in a special way), strain can be calculated easily from the following formula: $\varepsilon = \Delta R_c / (k \cdot R_c)$ (it is so called **inclination method**).

In other case we have to use so called **zero method**. When deformation occurs and active gauge's resistance changes, current starts flowing through BD branch and galvanometer indicates value different than 0 which is proportional to strain (this relation is derived later). Then by changing resistance R_1 (which can be regulated) one should set such value R_1' for which bridge circuit is in equilibrium state again. One can now find value of strain gauge's resistance before and after deformation and the difference between them (resistance increment):

$$R_c = R_k \frac{R_1}{R_2} \qquad R_c' = R_k \frac{R_1'}{R_2} \qquad \Rightarrow \qquad \Delta R_c = R_c' - R_c = \frac{R_k}{R_2}(R_1' - R_1)$$

Finally we obtain:

$$\varepsilon = \frac{R_1' - R_1}{R_1} \frac{1}{k}$$

Usually device regulating the resistance R_1 is scaled in such way that strain can be calculated directly from the results obtained and read from the indicator:

$$\varepsilon = \frac{2{,}0}{k_s}(M' - M) \cdot 10^3$$

M', M – results read from the indicator corresponding with two bridge circuit equilibrium states before and after deformation

k_s – constant parameter characteristic for scale used in measurement. The most convenient way is to set the value of k_s equal 2,0.

Relation between voltage (potential difference) change in BD branch and strain value is linear. Generally in a bridge circuit in arbitrary (not necessary equilibrium) state, potential difference is equal:

$$U_{BD} = J_2 R_c - J_1 R_1$$

In case of equilibrium state we can consider each pair of resistors as one (series circuit) and values of current flowing in both branches are equal:

$$J_1 = \frac{U}{R_1 + R_2} \qquad J_2 = \frac{U}{R_c + R_k}$$

Substituting those relations to the equation written before we obtain:

$$U_{BD}=\frac{U\,R_c}{R_c+R_k}+\frac{U\,R_1}{R_1+R_2}$$

We can differentiate it with respect to R_c and calculate finite increments:

$$\mathrm{d}\,U_{BD}=U\,\frac{R_k}{(R_c+R_k)^2}\,\mathrm{d}\,R_c \qquad \Rightarrow \qquad \Delta\,U_{BD}=U\,\frac{R_k}{(R_c+R_k)^2}\,\Delta\,R_c$$

Since $\Delta\,R_c=k\,\varepsilon\,R_c$ and $R_c=R_k$ thus:

$$\Delta\,U_{BD}=U\,\frac{R_k\,R_c}{(R_c+R_k)^2}\,k\,\varepsilon=\frac{U}{4}\,k\,\varepsilon$$

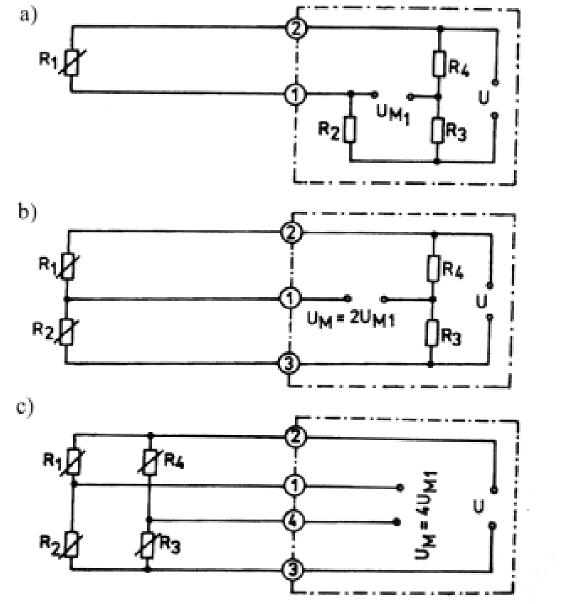
One can easily notice that when $R_c=R_k$ then voltage change before and after deformation does not depend on the value of resistances and it is proportional to the strain – this is why voltage measurement is can give us direct information about deformation.

STRAIN GAUGE COMPENSATION AND SELF-COMPENSATION

In some cases an additional deformation of the gauge occurs which is not related with external loading of the specimen and stresses appearing inside it but only with natural property of every body – namely – thermal deformation. Such deformation changes resistance of strain gauge. This change is registered by galvanometer thus it is a source of errors in measurement since thermal expansion properties of examined bodies and the material of which gauge is made are usually completely different. The simplest way of reducing this error practically to zero is **thermal compensation**. Any change in **active** (used for measurement) **strain gauge** resistance can be reduced by an identical resistance change in **compensating** (“dummy”) **strain gauge** – ratio of both resistances does not change in such case and the bridge circuit is still in an equilibrium state. It is possible only when:

- both, active and compensating, strain gauges are identical (the same gauge length, resistance and sensitivity coefficient)
- both strain gauges should be fastened using the same glue and compensating gauge should be glued to the plate made of the same material as the examined one, however that plate should not be loaded.
- both strain gauges should be close one to another.

Another way of reducing the error caused by thermal expansion is using **self-compensating gauges**. They should be used when using whole circuit with “dummy” gauges is not possible or when temperature gradient / variation is very large (compensating gauge deforms then in a different way than the active one). Self-compensating gauges are usually made of materials exhibiting very low thermal strain (i.e. constantan).



QURTER-, HALF- AND FULL-BRIDGE CIRCUITS

General schemes of quarter- , half- and full-bridge circuits are shown on the left. The main difference is number of active strain gauges used in measurements. Typical bridge circuit (described above) is the one with only one active strain gauge – it is called **quarter-bridge (**a). When two active gauges are used (b) it is called **half-bridge**, and in case of four gauges (c) – **full-bridge**. Sensitivity of half- and full-bridge circuits are respectively two and four times greater than sensitivity of quarter-bridge circuit (voltage change observed after deformation is two / four times greater).

UNIFORM STRESS STATE ANALYSIS

In case of uniform stress state (uniform tension or compression) direction of principal stresses in the body are parallel to the direction of loading. In any other direction absolute value of normal stress is always smaller than the principal one and it is a function of value of principal stress and angle between direction of loading and direction of the normal stress:

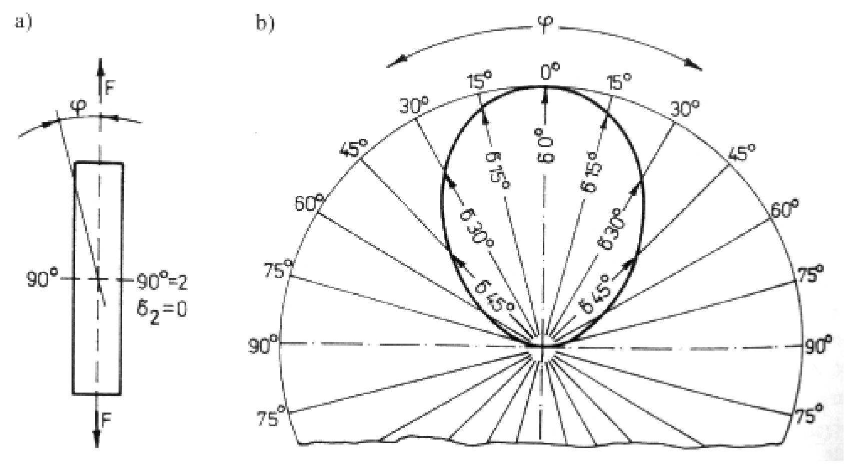
$$\mathbf{R}(\varphi)=\begin{bmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix} \qquad \boldsymbol{\sigma}=\begin{bmatrix}\sigma & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

σ – stress tensor
R – rotation matrix

Stress tensor coordinates in arbitrary rotated coordinate system:

$$\tilde{\boldsymbol{\sigma}}=\mathbf{R}^{\varphi}\cdot\boldsymbol{\sigma}\cdot\left[\mathbf{R}^{\varphi}\right]^{\mathrm{T}}=\begin{bmatrix}\sigma\cos^2\varphi & \sigma\cos\varphi\sin\varphi & 0\\ \sigma\cos\varphi\sin\varphi & \sigma\sin^2\varphi & 0\\ 0 & 0 & 0\end{bmatrix} \quad\Rightarrow\quad \sigma_{11}(\varphi)=\sigma\cos^2\varphi=\frac{1}{2}\sigma\left(1+\cos 2\varphi\right)$$

To determine value of normal stress in arbitrary chosen direction two strain gauges should be placed on the bar (active and compensating one – along the direction of loading). After measuring principal strain along the bar and calculating stress, normal stress in any other direction can be calculated from the formula above. It can be described by so called **polar diagram** shown below:

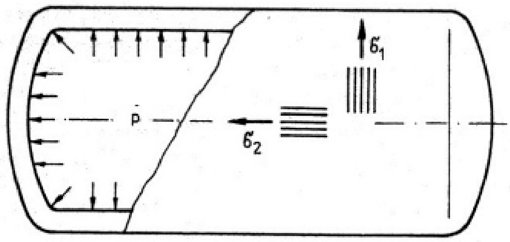


PLANE STRESS-STRAIN STATE ANALYSIS

Plane stress-strain state is a bit more difficult to analyze – this is due to fact that each longitudinal stress causes both longitudinal and transverse strains. If directions of principal strains and stresses are known (principal directions of stress and strain tensors are the same and principal directions of any symmetric tensor are always perpendicular one to another), as in case of i.e. cylindrical container loaded by internal pressure – see figure below) it is easy to determine value of stresses using generalized Hooke's Law:

$$\sigma_1 = \frac{E}{1-\nu^2}(\varepsilon_1 + \nu\,\varepsilon_2)$$

$$\sigma_2 = \frac{E}{1-\nu^2}(\varepsilon_2 + \nu\,\varepsilon_1)$$

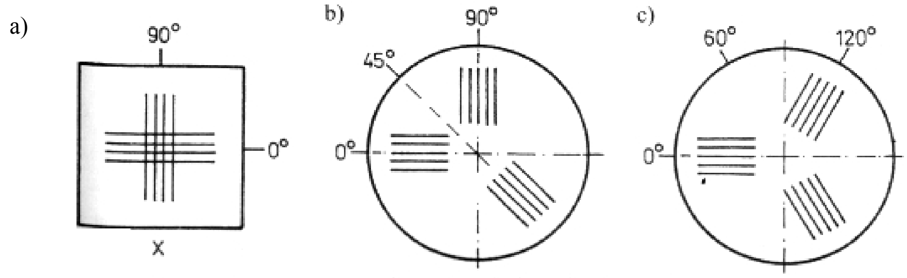


Where:

E, ν – Young modulus and Poisson's ratio

$\varepsilon_1, \varepsilon_2$ - strains measured

In fact in most of cases of measurements directions of principal stresses are unknown. Plane stress or strain state has only three independent components (two normal stresses and one shearing or two elongations and one distortion) – this is why we have to make not less and not more than three measurements but in three independent (non-parallel) directions. To do such measurement strain gauge rosettes are used – typical rosettes are shown below



a) two-gauge rosette b) perpendicular rosette (0°/45°/90°) c) delta type rosette (0°/60°/120°)

Neglecting strains perpendicular to the analyzed plane (which, under in-plane rotation, do not affect values of strain in the plane) we can write:

$$\mathbf{R}(\varphi) = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \qquad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{xy} & \varepsilon_{yy} \end{bmatrix}$$

In case of perpendicular rosette we obtain:

$$\boldsymbol{\varepsilon}^{(0°)} = \mathbf{R}^{0°} \cdot \boldsymbol{\varepsilon} \cdot \left[\mathbf{R}^{0°}\right]^{\mathrm{T}} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{xy} & \varepsilon_{yy} \end{bmatrix} = \begin{bmatrix} \varepsilon_0 & ? \\ ? & ? \end{bmatrix}$$

$$\boldsymbol{\varepsilon}^{(45°)} = \mathbf{R}^{45°} \cdot \boldsymbol{\varepsilon} \cdot \left[\mathbf{R}^{45°}\right]^{\mathrm{T}} = \begin{bmatrix} \frac{1}{2}(\varepsilon_{xx}+\varepsilon_{yy})-\varepsilon_{xy} & \varepsilon_{xx}-\varepsilon_{yy} \\ \varepsilon_{xx}-\varepsilon_{yy} & \frac{1}{2}(\varepsilon_{xx}+\varepsilon_{yy})+\varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} \varepsilon_{45} & ? \\ ? & ? \end{bmatrix}$$

$$\boldsymbol{\varepsilon}^{(90°)} = \mathbf{R}^{90°} \cdot \boldsymbol{\varepsilon} \cdot \left[\mathbf{R}^{90°}\right]^{\mathrm{T}} = \begin{bmatrix} \varepsilon_{yy} & -\varepsilon_{xy} \\ -\varepsilon_{xy} & \varepsilon_{xx} \end{bmatrix} = \begin{bmatrix} \varepsilon_{90} & ? \\ ? & ? \end{bmatrix}$$

Where $\varepsilon_0, \varepsilon_{45}, \varepsilon_{90}$ are values of strain indicated by gauges.

We obtain following system of equations:

$$\begin{cases} \varepsilon_{xx}=\varepsilon_0 \\ \varepsilon_{yy}=\varepsilon_{90} \\ \frac{1}{2}(\varepsilon_{xx}+\varepsilon_{yy})-\varepsilon_{xy}=\varepsilon_{45} \end{cases} \Rightarrow \begin{cases} \varepsilon_{xx}=\varepsilon_0 \\ \varepsilon_{yy}=\varepsilon_{90} \\ \varepsilon_{xy}=\frac{1}{2}(\varepsilon_0+\varepsilon_{90})-\varepsilon_{45} \end{cases}$$

knowing all components of strain tensor we can calculate its principal values and then principal stresses:

$$\varepsilon_{1,2}=\frac{\varepsilon_{xx}+\varepsilon_{yy}}{2}\pm\sqrt{\frac{(\varepsilon_{xx}-\varepsilon_{yy})^2}{2}+\varepsilon_{xy}^2}$$

and using generalized Hooke's Law we can find values of principal stresses.

Finally we obtain:

$$\sigma_{1,2}=\frac{E}{1-\nu^2}\,\frac{\varepsilon_0+\varepsilon_{90}}{2}\pm\frac{E}{\sqrt{2}(1+\nu)}\sqrt{(\varepsilon_0-\varepsilon_{45})^2+(\varepsilon_{90}-\varepsilon_{45})^2}$$

and the angle between direction of greater principal stress σ_1 and he direction of rosette 0° is given by following formula:

$$\varphi=\operatorname{arctg}\left[\frac{2\varepsilon_{45}-(\varepsilon_0+\varepsilon_{90})}{\varepsilon_0-\varepsilon_{90}}\right]$$

In case of delta-shaped rosette above relations has following form:

$$\sigma_{1,2}=\frac{E}{1-\nu^2}\,\frac{\varepsilon_0+\varepsilon_{60}+\varepsilon_{120}}{3}\pm\frac{E}{(1+\nu)}\sqrt{\left(\frac{2\varepsilon_0-\varepsilon_{60}-\varepsilon_{120}}{3}\right)^2+\frac{1}{3}(\varepsilon_{60}-\varepsilon_{120})^2}$$

$$\varphi=\operatorname{arctg}\left[\frac{\sqrt{3}(\varepsilon_{60}-\varepsilon_{120})}{2\varepsilon_0-\varepsilon_{60}-\varepsilon_{120}}\right]$$

VERIFICATION OF ASSUMED CONSTRUCTION MODEL

It is obvious that any performed engineering calculation is only an estimation or prediction of true behavior of real construction. Whole linear theory of elasticity bases on very specific assumptions which are sometimes not fulfilled in case of real structures and many further assumptions are necessary to make the theory applicable in practical calculation (i.e. small strains and displacements, ideal isotropy and homogeneity, linear stress-strain relation, ideal geometry, supporting and loading of a body and many others). Even without those additional assumptions, solving strongly complicated problems using Finite Element Method (or any other numerical methods) gives us solution which can still be different then it is in the reality. Strain gauge measurements help us in verifying whether assumed model (or theory used) is correct or not.

There is a simple way of determining normal stress distribution and finding values of internal forces in chosen cross-section of a bar. Stress vectors form a plane which is rotated and translated referring to the plane of chosen cross-section. To determine such plane we need three points (values of stresses in three different points of cross-section which do not lay on one straight line). Those stresses $\sigma_A, \sigma_B, \sigma_C$ can be measured easily using three strain gauges glued in proper places. We can calculate internal forces using commonly known relation between those internal forces and normal stresses and solving relatively simple system of linear equations

$$\begin{cases} \sigma_A = \dfrac{N}{A} + \dfrac{M_x}{I_x} y_A + \dfrac{M_y}{I_y} x_A \\ \sigma_B = \dfrac{N}{A} + \dfrac{M_x}{I_x} y_B + \dfrac{M_y}{I_y} x_B \\ \sigma_C = \dfrac{N}{A} + \dfrac{M_x}{I_x} y_C + \dfrac{M_y}{I_y} x_C \end{cases} \Rightarrow \begin{cases} N = A \cdot \dfrac{\sigma_A (x_C y_B - x_B y_C) + \sigma_B (x_A y_C - x_C y_A) + \sigma_C (x_B y_A - x_A y_B)}{x_A (y_C - y_B) + x_B (y_A - y_C) + x_C (y_B - y_A)} \\ M_x = I_x \cdot \dfrac{\sigma_A (x_B - x_C) + \sigma_B (x_C - x_A) + \sigma_C (x_A - x_B)}{x_A (y_C - y_B) + x_B (y_A - y_C) + x_C (y_B - y_A)} \\ M_y = I_y \cdot \dfrac{\sigma_A (y_C - y_B) + \sigma_B (y_A - y_C) + \sigma_C (y_B - y_A)}{x_A (y_C - y_B) + x_B (y_A - y_C) + x_C (y_B - y_A)} \end{cases}$$

Where:

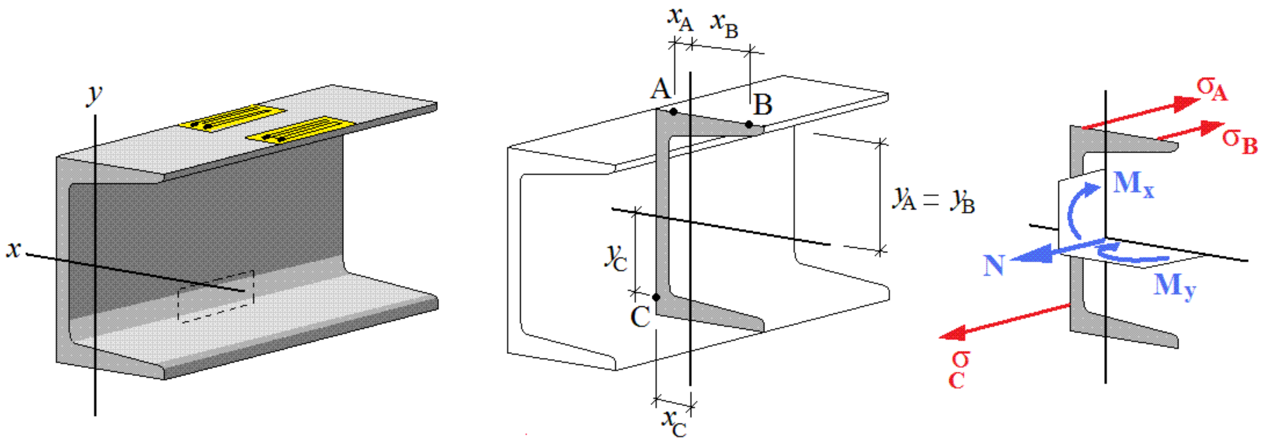
N, M_x, M_y - values of internal forces – axial force and bending moment respectively

σ_P - value of normal stress at point $P = A, B, C$

x_P, y_P - coordinates of point $P = A, B, C$ in coordinate system of principal central axes of inertia

A, I_x, I_y - area of cross-section, moments of inertia of cross-section

Above relations are called Aistow equations.



Trusses are good examples of construction that behaves in reality quite different than it is assumed in model. Truss should be loaded only in nodes and all bars' connections are joints – thus in bars of theoretical truss only axial forces should occur. In fact bar connections are much more stiff, they are always welded or bolted. Rotational deformation in nodes is thus blocked to some extent and that is the reason of bending moment appearing in the bar. Another thing is that dead weight load (weight of construction itself) cannot be applied to the structure in nodes – it also causes occurrence of bending moments.