

Lecture 2.

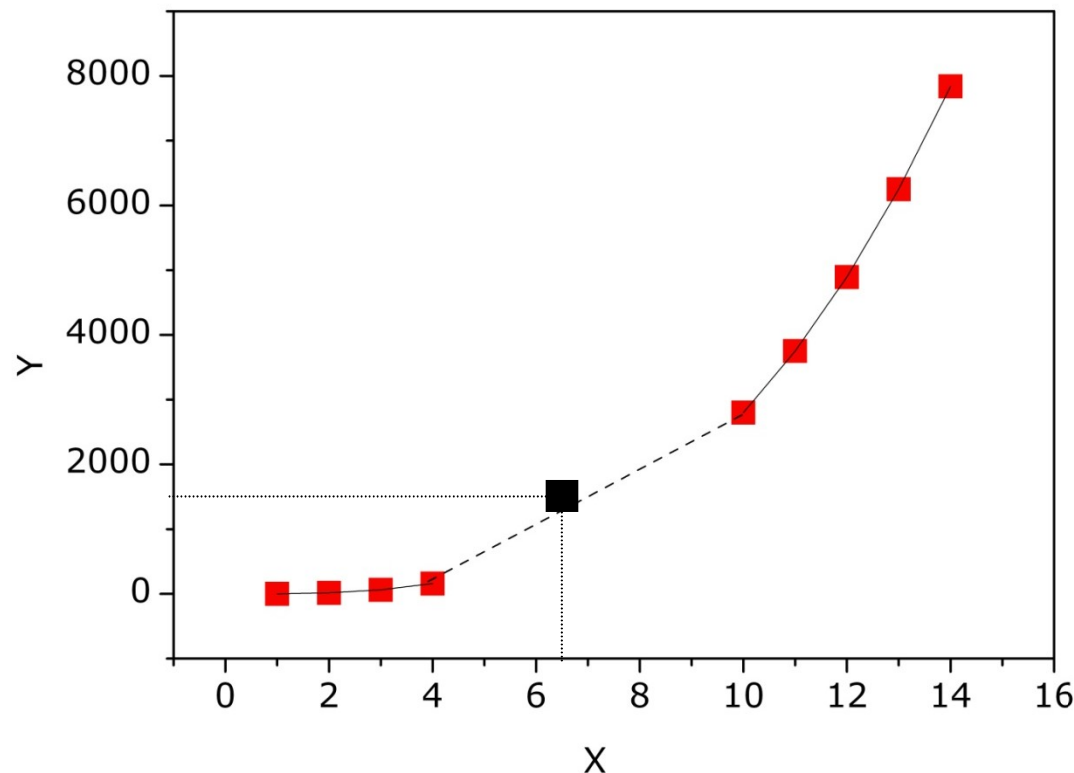
Approximation vs. interpolation

Outline

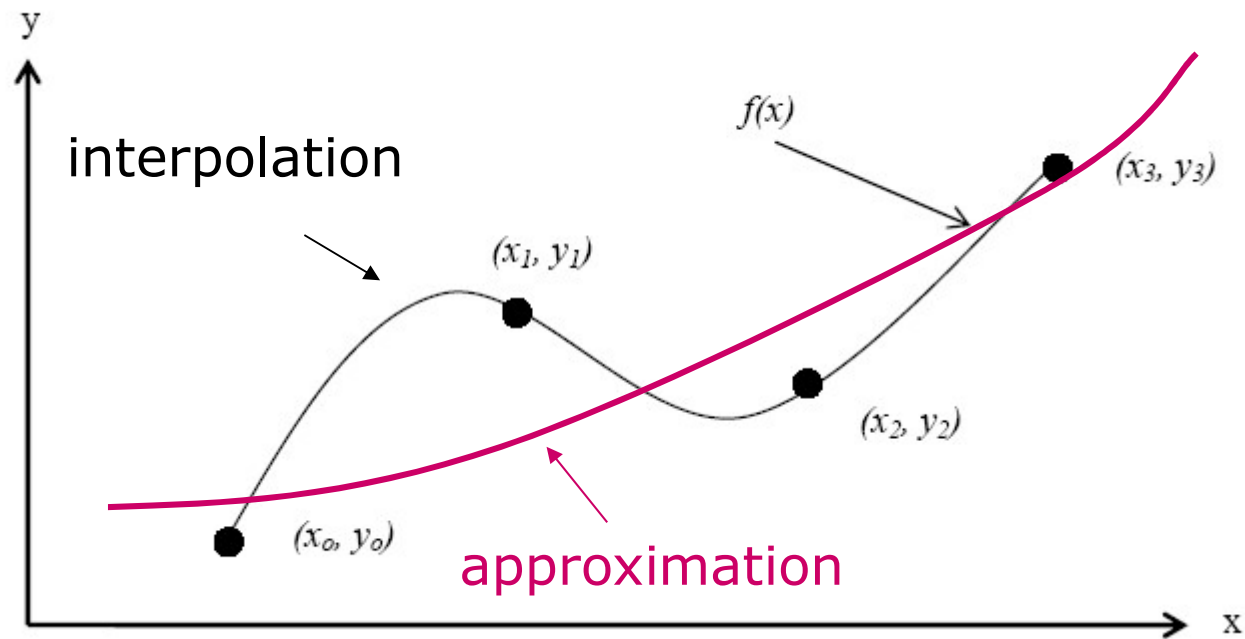
- Approximation
- Polynomial interpolation
- Examples

Principle of interpolation

Consider discrete points $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$.
How does one find the value of y at any other value of x ?



Approximation vs. interpolation



Approximation

We would like to approximate $f(x)$ by linear combination of functions that belong to a certain, particular class .

$\{x^n\}$ ($n = 0, 1, \dots$) like in Taylor's series

$\{p_n(x)\}$ ($n = 0, 1, \dots$) $p_n(x)$ is a polynomial of order n

$\{\sin(nx), \cos(nx)\}$ ($n = 0, 1, 2, \dots$) trigonometric polynomials

The most important is a polynomial interpolation

Approximation

Linear approximation $f(x)$

$$f(x) \approx a_0 g_0(x) + a_1 g_1(x) + \dots + a_m g_m(x)$$

$$\{g_n(x)\} \quad (n = 0, 1, \dots)$$

Constant coefficients: $a_i \quad (i = 0, 1, \dots, m)$

Linear approximation is widely used, because non-linear approximation is very complex and difficult.

Sometimes rational function approximation is better:

$$f(x) \approx \frac{a_0 g_0(x) + a_1 g_1(x) + \dots + a_m g_m(x)}{b_0 g_0(x) + b_1 g_1(x) + \dots + b_k g_k(x)}$$

Approximation

The selection criteria for constant coefficients

$$a_i (i = 0, 1, \dots, m)$$

There exist three types of approximations of major importance:

- **interpolation**

coefficients are chosen, so that at the points

$$x_i (i = 1, 2, \dots, p)$$

the approximate function with its first derivatives is equal with $f(x)$ and its derivatives (with the accuracy of rounding errors)

Approximation

The selection criteria for constant coefficients $a_i (i = 0, 1, \dots, m)$

- **mean square approximation**

We want to find a minimum of integral of the square of the difference between $f(x)$ and its approximation in interval $\langle x_1, x_2 \rangle$ or weighted sum of squared errors in collection of discrete points within the interval $\langle x_1, x_2 \rangle$

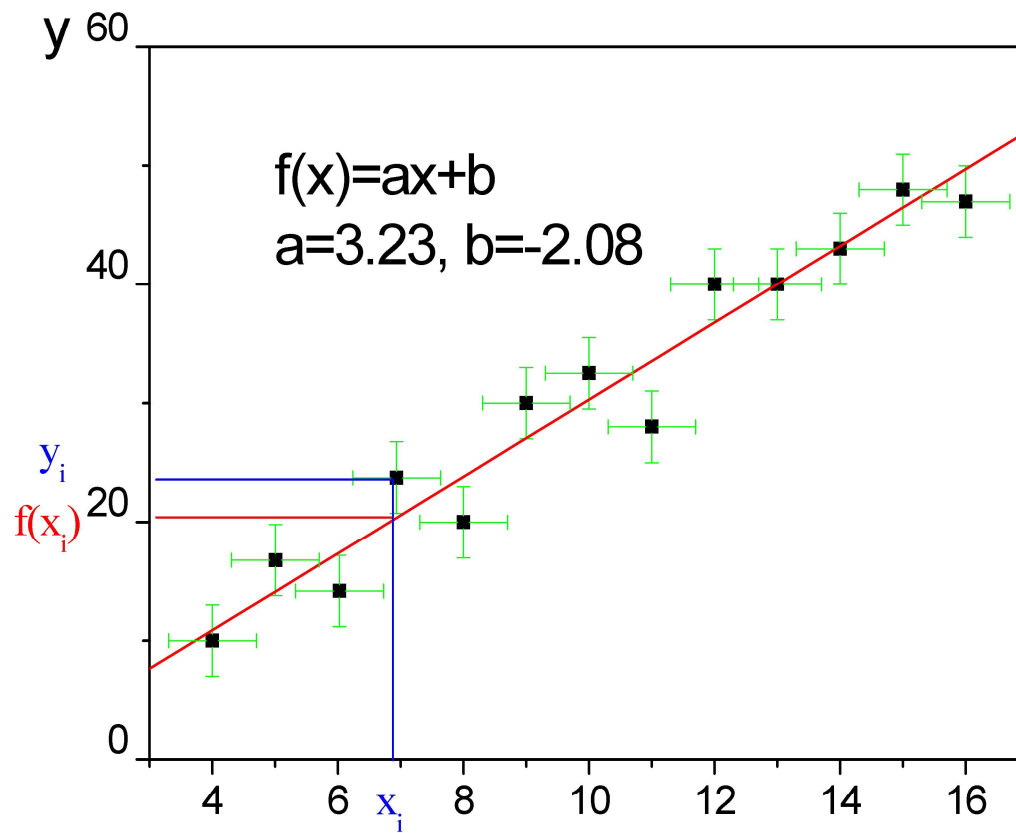
- **uniform approximation**

finding the maximum of the smallest difference between $f(x)$ and its approximation within the interval $\langle x_1, x_2 \rangle$

The least squares methods

Linear regression

$$S^2 = \sum_i^n [y_i - (ax_i + b)]^2 = \min$$



The least squares criterion:

$$\frac{\partial S^2}{\partial a} = 0 \quad \frac{\partial S^2}{\partial b} = 0$$

Linear equations for a and b

$$a \sum x_i^2 + b \sum x_i = \sum x_i y_i$$

$$a \sum x_i + b n = \sum y_i$$

Solving the above set of equations gives a and b :

$$a = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{W}$$

$$b = \frac{\sum x_i^2 \sum y_i - \sum x_i \sum x_i y_i}{W}$$

where: the main determinant W is given by:

$$W = n \sum x_i^2 - \left(\sum x_i \right)^2$$

Standard deviation for $u(a)$ and $u(b)$:

$$u(a) = \sqrt{\frac{n}{n-2}} \sqrt{\frac{S^2}{W}}$$

$$u(b) = u(a) \sqrt{\frac{\sum x_i^2}{n}}$$

Polynomial approximation

A polynomial is a common choice for an interpolating function because polynomials are easy to:

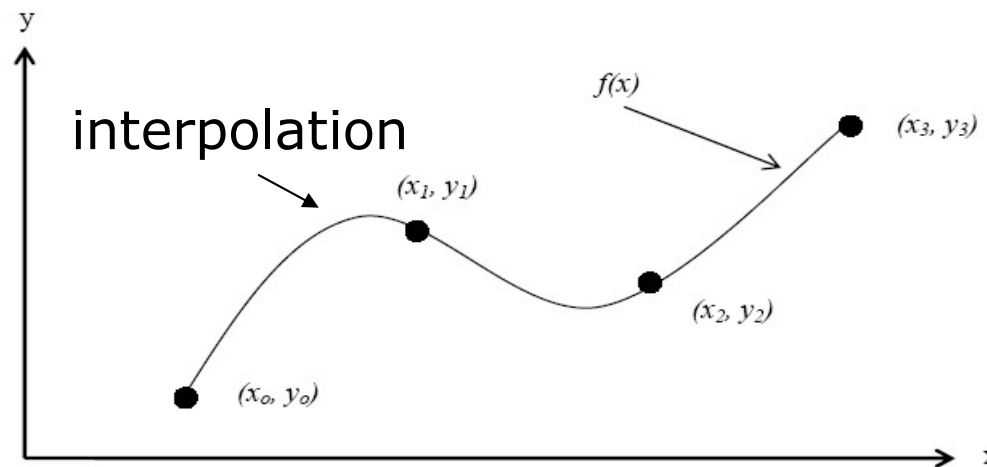
- evaluate
- differentiate
- integrate

Polynomial interpolation

Assumption:

In the $[a, b]$ interval are $(n + 1)$ different points $x_0, x_1, \dots, x_n$, called interpolation nodes, and the value of a function $y = f(x)$ at these points:

$$f(x_i) = y_i \text{ dla } i = 0, 1, \dots, n.$$



Polynomial interpolation

Main goal of interpolation:

Determination of the approximate value of the function at points which are not nodes and estimation of interpolation error

1. We want to find $F(x)$ which approximates function $f(x)$ in the $[a,b]$ interval.
2. The function $F(x)$ in the nodes of interpolation has the same value as a function $y = f(x)$.
3. There is a unique polynomial of degree less than or equal to n passing through $n+1$ given points

Interpolation – direct method

Polynomial interpolation involves finding a polynomial of order n that passes through the $n+1$ $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$ points

$$y = a_0 + a_1x + \dots + a_nx^n.$$

where $a_0, a_1, \dots, a_n$ are constant coefficients (R)

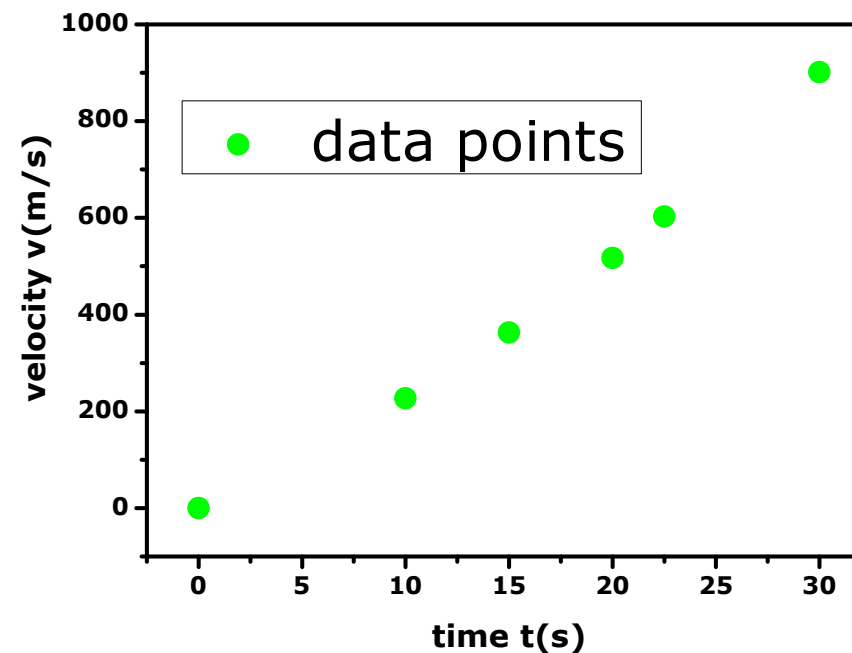
- Consider $n+1$ equations to find $n+1$ constant
- Use x value in polynomial to find y value

Example



Table 1 Velocity v as a function of time t

$t(s)$	$v(m/s)$
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67



Determine the value of the velocity at $t = 16$ s using the direct method of interpolation

Linear interpolation

$$v(t) = a_0 + a_1 t$$

$$v(15) = a_0 + a_1(15) = 362.78$$

$$v(20) = a_0 + a_1(20) = 517.35$$

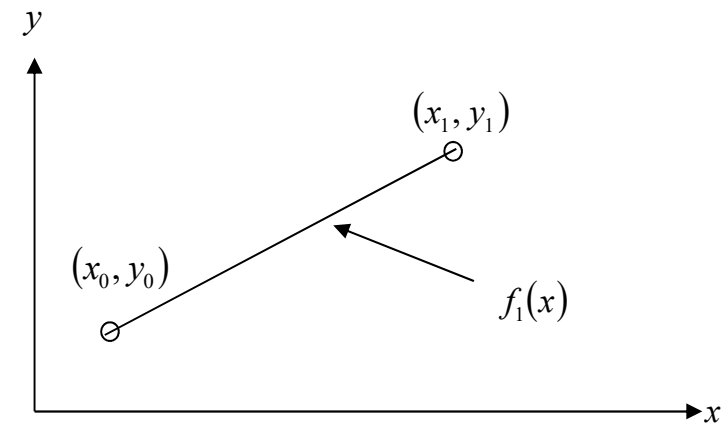
The solution of equations

$$a_0 = -100.93$$

$$a_1 = 30.914$$

So: $v(t) = -100.93 + 30.914t, 15 \leq t \leq 20$

$$v(16) = -100.93 + 30.914(16) = 393.7 \text{ m/s}$$



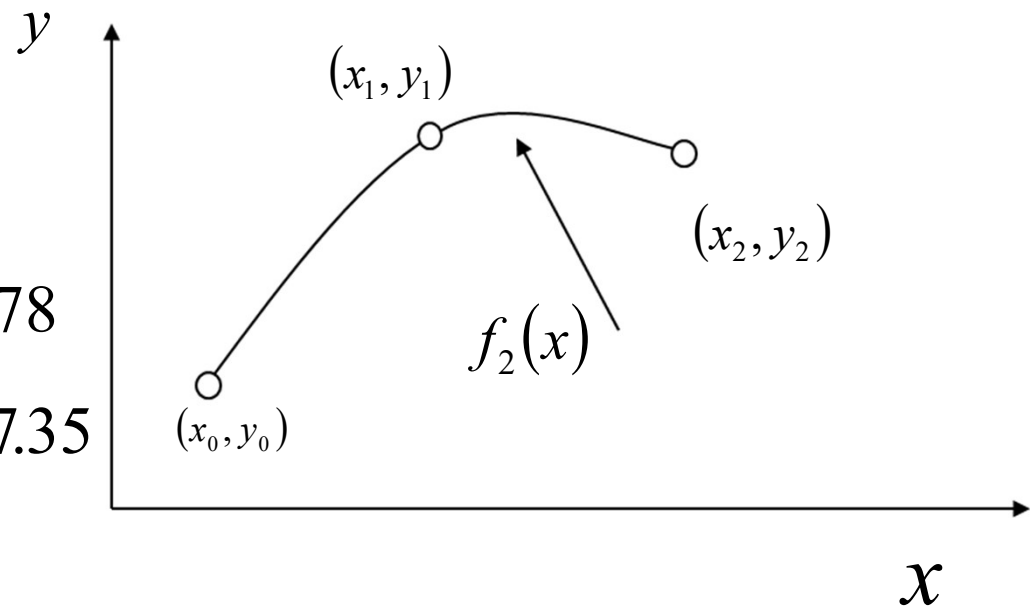
Quadratic interpolation

$$v(t) = a_0 + a_1 t + a_2 t^2$$

$$v(10) = a_0 + a_1(10) + a_2(10)^2 = 227.04$$

$$v(15) = a_0 + a_1(15) + a_2(15)^2 = 362.78$$

$$v(20) = a_0 + a_1(20) + a_2(20)^2 = 517.35$$



The solution of equations

$$a_0 = 12.05 \quad a_1 = 17.733 \quad a_2 = 0.3766$$

$$v(t) = 12.05 + 17.733t + 0.3766t^2, \quad 10 \leq t \leq 20$$

$$v(16) = 12.05 + 17.733(16) + 0.3766(16)^2 = 392.19 \text{ m/s}$$

Quadratic interpolation

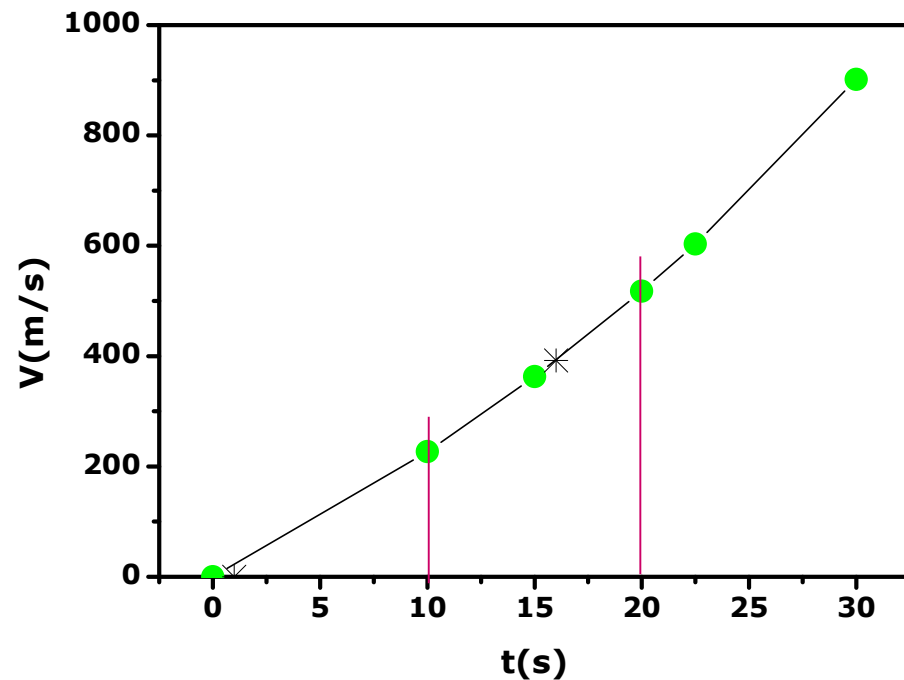
$$v(t) = 12.05 + 17.733t + 0.3766t^2, \quad 10 \leq t \leq 20$$

$$v(16) = 392.19 \text{ m/s}$$

Relative error:

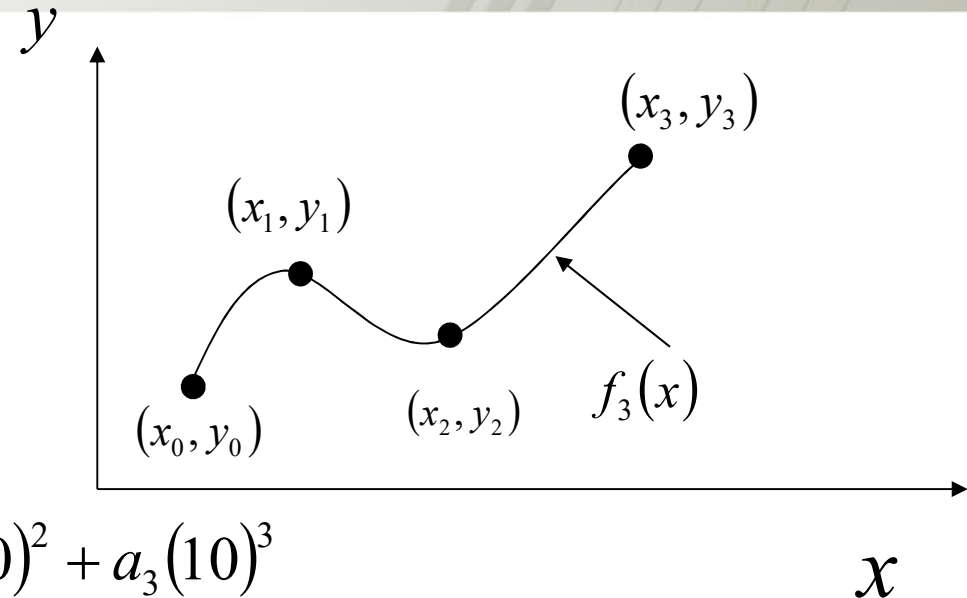
$$|\epsilon_a| = \left| \frac{392.19 - 393.70}{392.19} \right| \times 100$$

$$= 0.38410\%$$



Cubic interpolation

$$v(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$



$$v(10) = 227.04 = a_0 + a_1(10) + a_2(10)^2 + a_3(10)^3$$

$$v(15) = 362.78 = a_0 + a_1(15) + a_2(15)^2 + a_3(15)^3$$

$$v(20) = 517.35 = a_0 + a_1(20) + a_2(20)^2 + a_3(20)^3$$

$$v(22.5) = 602.97 = a_0 + a_1(22.5) + a_2(22.5)^2 + a_3(22.5)^3$$

Cubic interpolation

Homework

The solution of equations:

$$\begin{cases} a_0 + 10a_1 + 100a_2 + 1000a_3 = 227.04 \\ a_0 + 15a_1 + 225a_2 + 3375a_3 = 362.78 \\ a_0 + 20a_1 + 400a_2 + 8000a_3 = 517.35 \\ a_0 + 22.5a_1 + 506.25a_2 + 11390.625a_3 = 602.97 \end{cases}$$

Calculate and draw $v(t)$

Cubic interpolation - solution

$$a_0 = -4.2540$$

$$a_1 = 21.266$$

$$a_2 = 0.13204$$

$$a_3 = 0.0054347$$

$$10 \leq t \leq 22.5$$

$$v(t) = -4.2540 + 21.266t + 0.13204t^2 + 0.0054347t^3,$$

$$v(16) = 392.06 \text{ m / s}$$

Relative error

$$\begin{aligned} |\epsilon_a| &= \left| \frac{392.06 - 392.19}{392.06} \right| \times 100 \\ &= 0.033269 \% \end{aligned}$$

Comparison

Polynomial order	1	2	3
$v(t = 16) \text{ m/s}$	393.7	392.19	392.06
Relative error	-----	0.38410 %	0.033269 %

Displacement

from $t=11\text{s}$ to $t=16\text{s}$

$$v(t) = -4.2540 + 21.266t + 0.13204t^2 + 0.0054347t^3, 10 \leq t \leq 22.5$$

$$s(16) - s(11) = \int_{11}^{16} v(t) dt$$

$$= \int_{11}^{16} (-4.2540 + 21.266t + 0.13204t^2 + 0.0054347t^3) dt$$

$$= \left[-4.2540t + 21.266 \frac{t^2}{2} + 0.13204 \frac{t^3}{3} + 0.0054347 \frac{t^4}{4} \right]_{11}^{16}$$

$$= 1605 \text{ m}$$

Acceleration

$$v(t) = -4.2540 + 21.266t + 0.13204t^2 + 0.0054347t^3, 10 \leq t \leq 22.5$$

$$\begin{aligned} a(t) &= \frac{d}{dt} v(t) \\ &= \frac{d}{dt} (-4.2540 + 21.266t + 0.13204t^2 + 0.0054347t^3) \\ &= 21.266 + 0.26408t + 0.016304t^2, \quad 10 \leq t \leq 22.5 \end{aligned}$$

$$\begin{aligned} a(16) &= 21.266 + 0.26408(16) + 0.016304(16)^2 \\ &= 29.665 \text{ m/s}^2 \end{aligned}$$

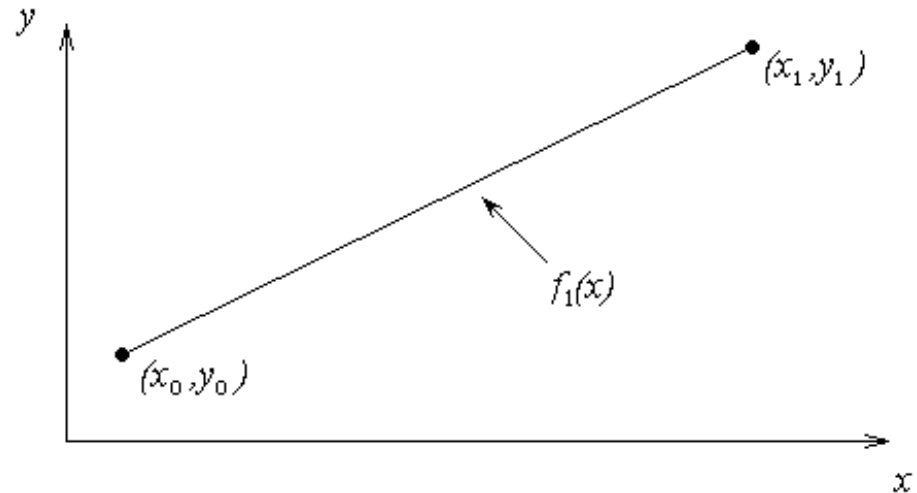
Newton's divided difference interpolation

Linear interpolation: data points (x_0, y_0) , (x_1, y_1) ,

Let's find:

$$f_1(x) = b_0 + b_1(x - x_0)$$

$$\begin{cases} b_0 = f(x_0) \\ b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \end{cases}$$

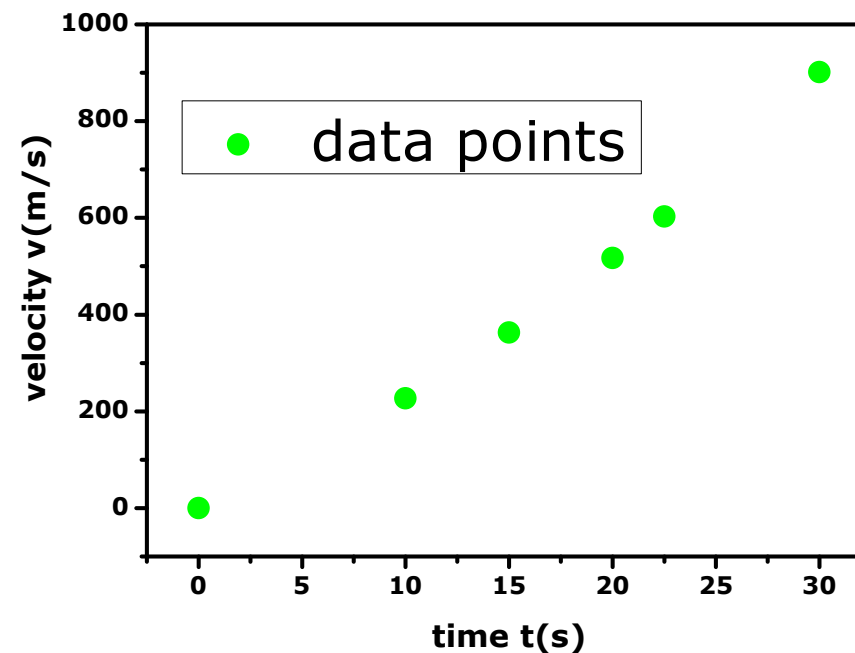


Example



Table 1 Velocity v as a time function t

$t(s)$	$v(m/s)$
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67



Determine the value of the velocity at time $t = 16$ s using Newton's method of interpolation

Linear interpolation

$$v(t) = b_0 + b_1(t - t_0)$$

It is known that :

$$t_0 = 15, \quad v(t_0) = 362.78$$

$$t_1 = 20, \quad v(t_1) = 517.35$$

We found:

$$b_0 = v(t_0) = 362.78$$

$$b_1 = \frac{v(t_1) - v(t_0)}{t_1 - t_0} = 30.914$$

So:

$$\begin{aligned} v(t) &= b_0 + b_1(t - t_0) = \\ &= 362.78 + 30.914(t - 15), \quad 15 \leq t \leq 20 \end{aligned}$$

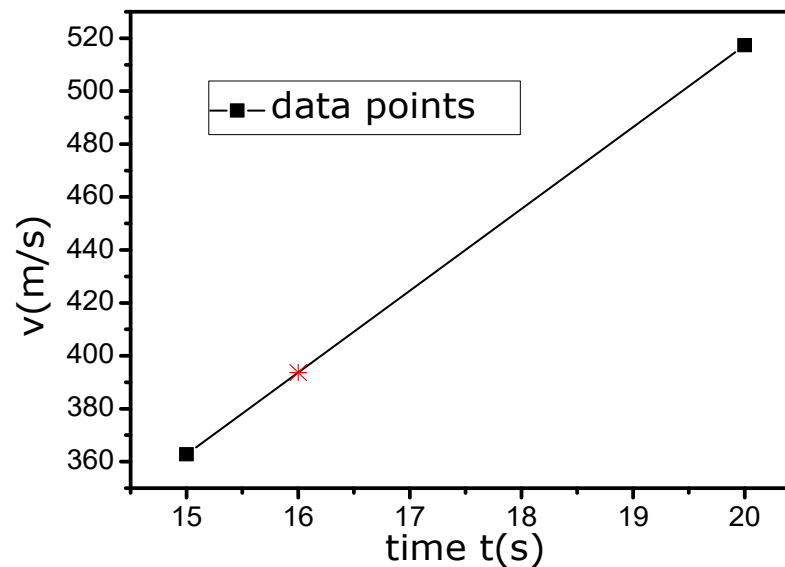
Homework: please compare obtained result with the direct method of interpolation

Linear interpolation

$$v(t) = b_0 + b_1(t - t_0)$$

Velocity at $t=16$ s is:

$$\begin{aligned} v(t) &= b_0 + b_1(t - t_0) = \\ &= 362.78 + 30.914(16 - 15) \\ &= 393.69 \text{ m/s} \end{aligned}$$

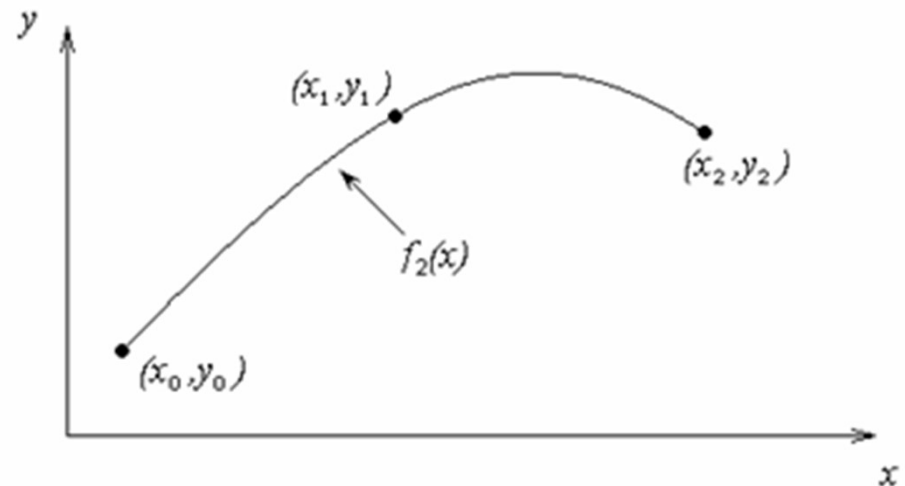


Quadratic interpolation

Consider the points: (x_0, y_0) , (x_1, y_1) , (x_2, y_2) ,
points:

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$\left\{ \begin{array}{l} b_0 = f(x_0) \\ b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \\ b_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0} \end{array} \right.$$



Quadratic interpolation

It is known that :

$$t_0 = 10, \quad v(t_0) = 227.04$$

$$t_1 = 15, \quad v(t_1) = 362.78$$

$$t_2 = 20, \quad v(t_2) = 517.35$$

We found:

$$b_0 = v(t_0) = 227.04$$

$$\begin{aligned} b_1 &= \frac{v(t_1) - v(t_0)}{t_1 - t_0} = \frac{362.78 - 227.04}{15 - 10} = \\ &= 27.148 \end{aligned}$$

$$\begin{aligned} b_2 &= \frac{\frac{v(t_2) - v(t_1)}{t_2 - t_1} - \frac{v(t_1) - v(t_0)}{t_1 - t_0}}{t_2 - t_0} = \frac{30.914 - 27.148}{10} = \\ &= 0.37660 \end{aligned}$$

Quadratic interpolation

So:

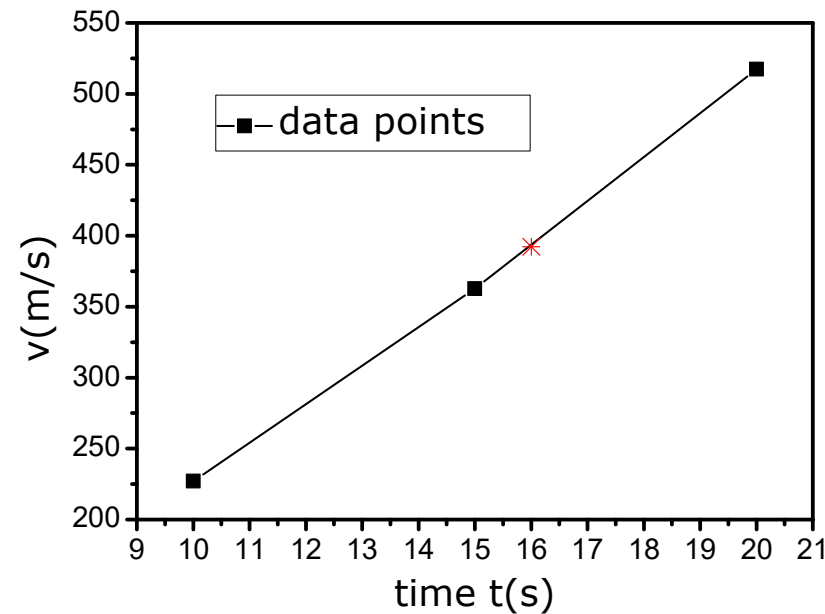
$$\begin{aligned}v(t) &= b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1) = \\&= 227.04 + 27.148(t - 10) + 0.37660(t - 10)(t - 15), \quad 10 \leq t \leq 20\end{aligned}$$

for $t=16s$:

$$\begin{aligned}v(16) &= b_0 + b_1(16 - t_0) + b_2(16 - t_0)(16 - t_1) = \\&= 227.04 + 27.148(16 - 10) + 0.37660(16 - 10)(16 - 15) \\&= 392.19 \text{ m/s}\end{aligned}$$

Homework: please compare obtained result with the direct method of interpolation

Quadratic interpolation



The relative error in comparison with the previous interpolation

$$\begin{aligned}
 |\epsilon_a| &= \left| \frac{392.19 - 393.69}{392.19} \right| \times 100 \\
 &= 0.38502 \%
 \end{aligned}$$

General formula

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

where

$$b_0 = f[x_0] = f(x_0)$$

$$b_1 = f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \quad \leftarrow \text{first order difference quotient}$$

$$b_2 = f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$



So:

second order difference quotient

$$f_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$

General formula

Consider $(n+1)$ points

$$(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$$

$$f_n(x) = b_0 + b_1(x - x_0) + \dots + b_n(x - x_0)(x - x_1)\dots(x - x_{n-1})$$

where

$$b_0 = f[x_0]$$

$$b_1 = f[x_1, x_0]$$

$$b_2 = f[x_2, x_1, x_0]$$

$$\vdots$$

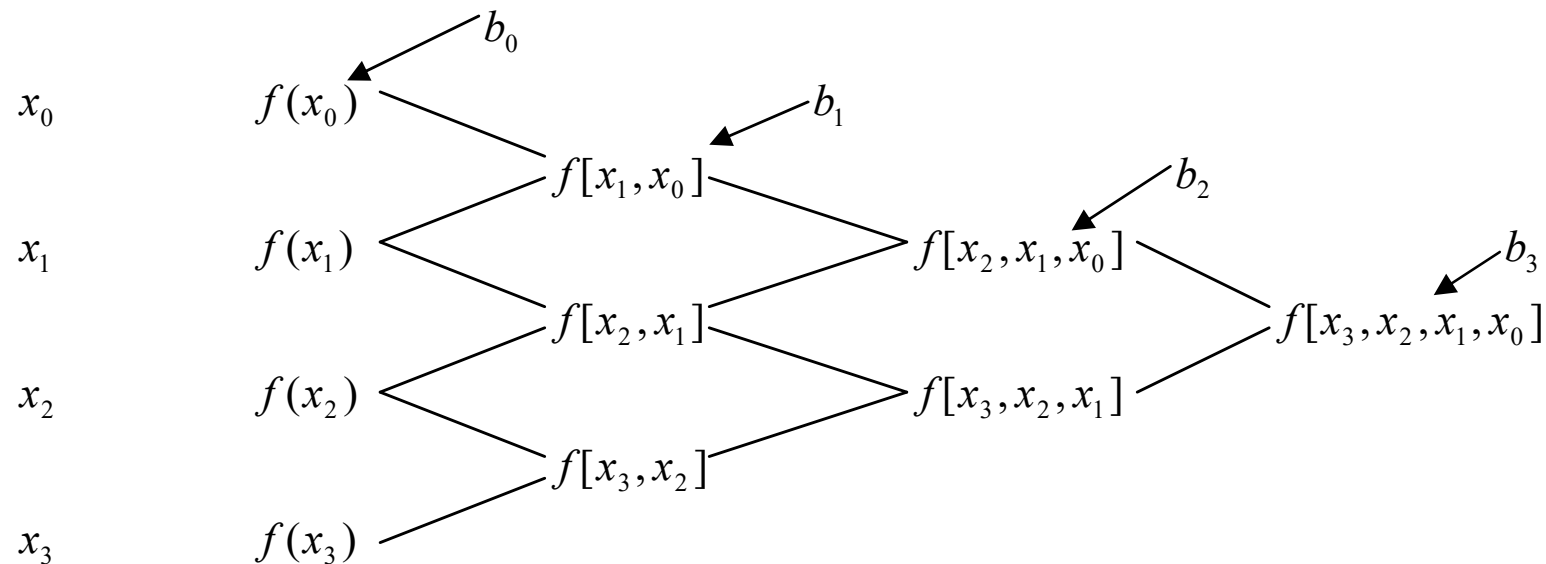
$$b_{n-1} = f[x_{n-1}, x_{n-2}, \dots, x_0]$$

$$b_n = f[x_n, x_{n-1}, \dots, x_0]$$

Cubic interpolation

3-rd degree polynomial, with data (x_0, y_0) , (x_1, y_1) , (x_2, y_2) , (x_3, y_3) , is:

$$f_3(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1) + f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)$$



Cubic interpolation

Homework

Find the equation and calculate the speed v at 16s. Use the Newton cubic method of interpolation :

$$v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1) + b_3(t - t_0)(t - t_1)(t - t_2)$$

Data

$$t_0 = 10, \quad v(t_0) = 227.04$$

$$t_1 = 15, \quad v(t_1) = 362.78$$

$$t_2 = 20, \quad v(t_2) = 517.35$$

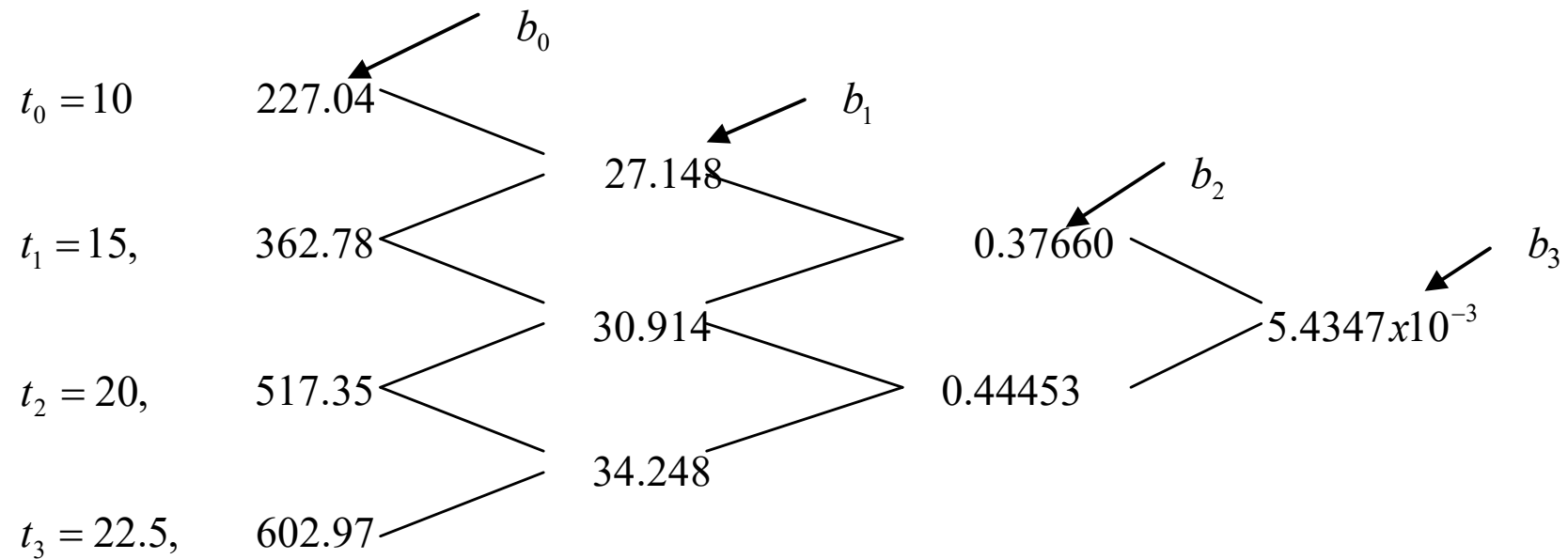
$$t_3 = 22.5, \quad v(t_3) = 602.97$$

Find the coefficients b_i

Find the distance covered by the rocket from $t = 11$ s to $t = 16$ s.

Find the acceleration at $t = 16$ s

Solution



$$b_0 = 227.04; \quad b_1 = 27.148; \quad b_2 = 0.37660; \quad b_3 = 5.4347 \times 10^{-3}$$

Comparison

Polynomial order	1	2	3
$v(t=16)$ m/s	393.69	392.19	392.06
The relative error of approximation	-----	0.38502 %	0.033427 %

Interpolation at equidistant points

Consider the function $f(x_i)=y_i$ for $i=0,1,...,n$ at equidistant points: $x_i = x_0 + ih$

The first Newton's polynomial interpolation is:

$$N_n^I(x) = y_0 + \frac{\Delta y_0}{1!h}(x - x_0) + \frac{\Delta^2 y_0}{2!h^2}(x - x_0)(x - x_1) + \dots + \frac{\Delta^n y_0}{n!h^n}(x - x_0)(x - x_1)\dots(x - x_{n-1})$$

where $\Delta^k f(x_0)$ is a progressive k-th order difference

$$\Delta f = f(x+h) - f(x)$$

$$\Delta^k f = \Delta(\Delta^{k-1} f)$$

Interpolation at equidistant points

Newton polynomial interpolation is preferred near beginning of the array. Near the end of the array we use

$$N_n^{II}(x) = y_n + \frac{\Delta y_n}{1!h}(x - x_n) + \frac{\Delta^2 y_n}{2!h^2}(x - x_n)(x - x_{n-1}) + \dots +$$
$$+ \frac{\Delta^n y_n}{n!h^n}(x - x_n)(x - x_{n-1})\dots(x - x_1)$$

Newton second polynomial interpolation with backward differences

$$\nabla y_i = \Delta y_{i-1} = f(x_i) - f(x_{i-1})$$

$$\nabla^k y_i = \Delta^k y_{i-k}$$

Forward difference

$$\Delta y_i = f(x_i + h) - f(x_i) = y_{i+1} - y_i$$

$$\Delta^2 y_i = \Delta(\Delta y_i) = \Delta y_{i+1} - \Delta y_i$$

Backward difference

$$\nabla y_i = f(x_i) - f(x_i - h) = y_i - y_{i-1}$$

$$\nabla^2 y_i = \nabla(\nabla y_i) = \nabla y_i - \nabla y_{i-1}$$

INTERPOLATION BY LAGRANGE POLYNOMIAL

$$W_n(x) = \sum_{j=0}^n f(x_j) \frac{(x-x_0)(x-x_1)\dots(x-x_{j-1})(x-x_{j+1})\dots(x-x_n)}{(x_j-x_0)(x_j-x_1)\dots(x_j-x_{j-1})(x_j-x_{j+1})\dots(x_j-x_n)}$$

General:

$$W_n(x) = \sum_{j=0}^n f(x_j) \frac{\omega_n(x)}{(x-x_j) \left\{ \frac{\omega_n(x)}{x-x_j} \right\} \Big|_{x=x_j}} = \sum_{j=0}^n f(x_j) \frac{\omega_n(x)}{(x-x_j) \omega'_n(x_j)}$$

where: $\omega_n(x) = (x-x_0)(x-x_1)\dots(x-x_n)$

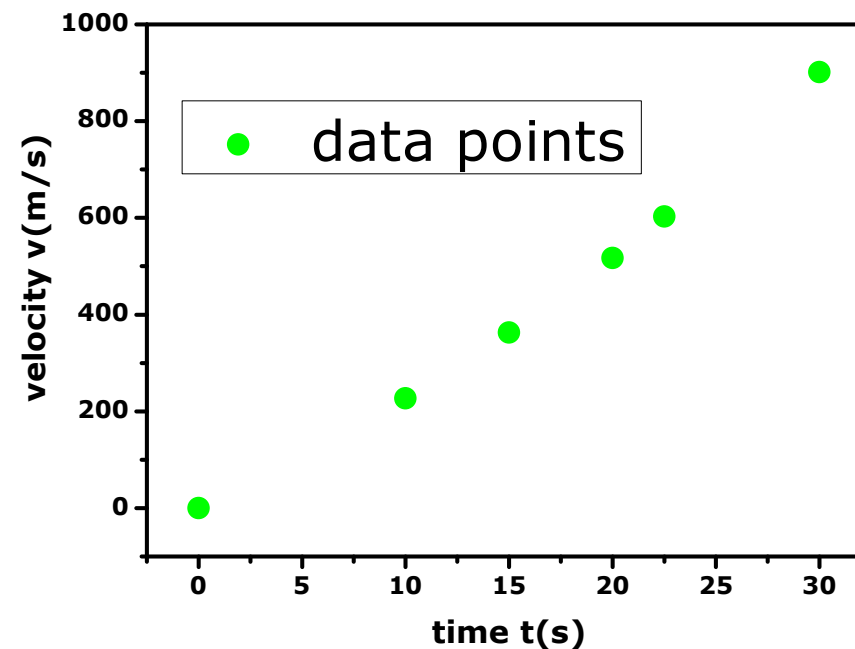
$\omega'_n(x_j)$ is the value of the derivative of a polynomial $\omega_n(x)$ at point x_j which is a root of the polynomial

Example



Table 1 Velocity v as a function of time t

$t(s)$	$v(m/s)$
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67



Determine the value of the velocity at $t = 16$ s using the Lagrange polynomial interpolation method

INTERPOLATION BY LAGRANGE POLYNOMIAL

$$v(t) = \sum_{i=0}^1 L_i(t)v(t_i) = L_0(t)v(t_0) + L_1(t)v(t_1)$$

It is known that:

$$t_0 = 15, \quad v(t_0) = 362.78$$

$$t_1 = 20, \quad v(t_1) = 517.35$$

We get:

$$L_0(t) = \prod_{\substack{j=0 \\ j \neq 0}}^1 \frac{t - t_j}{t_0 - t_j} = \frac{t - t_1}{t_0 - t_1}$$

$$L_1(t) = \prod_{\substack{j=0 \\ j \neq 1}}^1 \frac{t - t_j}{t_1 - t_j} = \frac{t - t_0}{t_1 - t_0}$$

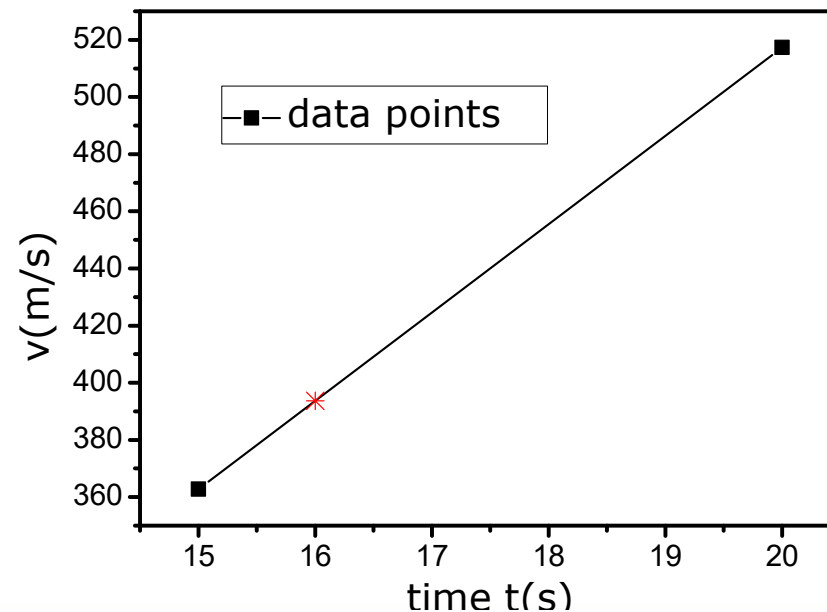
So:

$$\begin{aligned} v(t) &= \frac{t - t_1}{t_0 - t_1} v(t_0) + \frac{t - t_0}{t_1 - t_0} v(t_1) = \\ &= \frac{t - 20}{15 - 20} 362.78 + \frac{t - 15}{20 - 15} 517.35 \end{aligned}$$

Homework: please compare the obtained result with the direct method of interpolation

INTERPOLATION BY LAGRANGE POLYNOMIAL

$$\begin{aligned}v(16) &= \frac{16-20}{15-20} (362.78) + \frac{16-15}{20-15} (517.35) \\&= 0.8 (362.78) + 0.2 (517.35) = \\&= 393.7 \text{ m/s}\end{aligned}$$



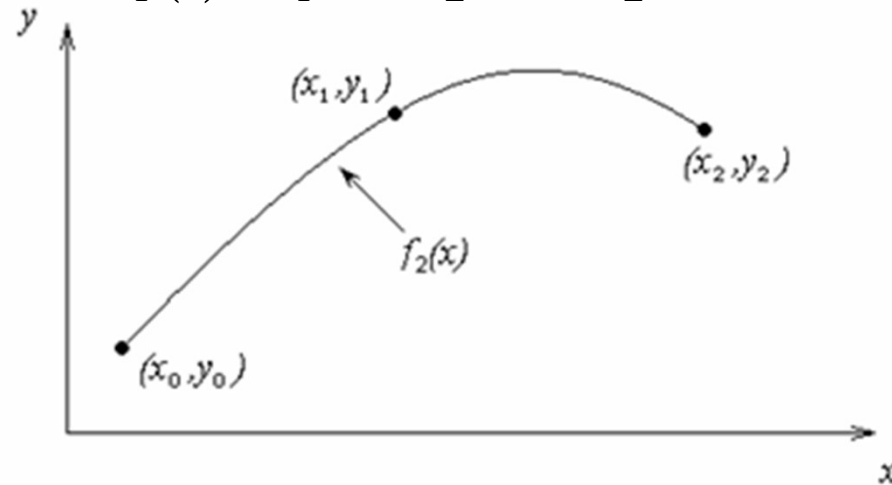
Quadratic interpolation

Consider the points: (x_0, y_0) , (x_1, y_1) , (x_2, y_2)

We want to find:

$$\begin{aligned}
 v(t) &= \sum_{i=0}^2 L_i(t) v(t_i) = \\
 &= L_0(t) v(t_0) + L_1(t) v(t_1) + L_2(t) v(t_2)
 \end{aligned}$$

$$L_i(t) = \prod_{\substack{j=0 \\ j \neq i}}^2 \frac{t - t_j}{t_i - t_j}$$



Quadratic interpolation

It is known that:

$$t_0 = 10, \quad v(t_0) = 227.04$$

$$t_1 = 15, \quad v(t_1) = 362.78$$

$$t_2 = 20, \quad v(t_2) = 517.35$$

We get:

$$L_0(t) = \prod_{\substack{j=0 \\ j \neq 0}}^2 \frac{t - t_j}{t_0 - t_j} = \frac{(t - t_1)(t - t_2)}{(t_0 - t_1)(t_0 - t_2)}$$

$$L_1(t) = \prod_{\substack{j=0 \\ j \neq 1}}^2 \frac{t - t_j}{t_1 - t_j} = \frac{(t - t_0)(t - t_2)}{(t_1 - t_0)(t_1 - t_2)}$$

$$L_2(t) = \prod_{\substack{j=0 \\ j \neq 2}}^2 \frac{t - t_j}{t_2 - t_j} = \frac{(t - t_0)(t - t_1)}{(t_2 - t_0)(t_2 - t_1)}$$

Then:

$$v(t) = \frac{t - t_1}{t_0 - t_1} \frac{t - t_2}{t_0 - t_2} v(t_0) + \frac{t - t_0}{t_1 - t_0} \frac{t - t_2}{t_1 - t_2} v(t_1) + \frac{t - t_0}{t_2 - t_0} \frac{t - t_1}{t_2 - t_1} v(t_2)$$

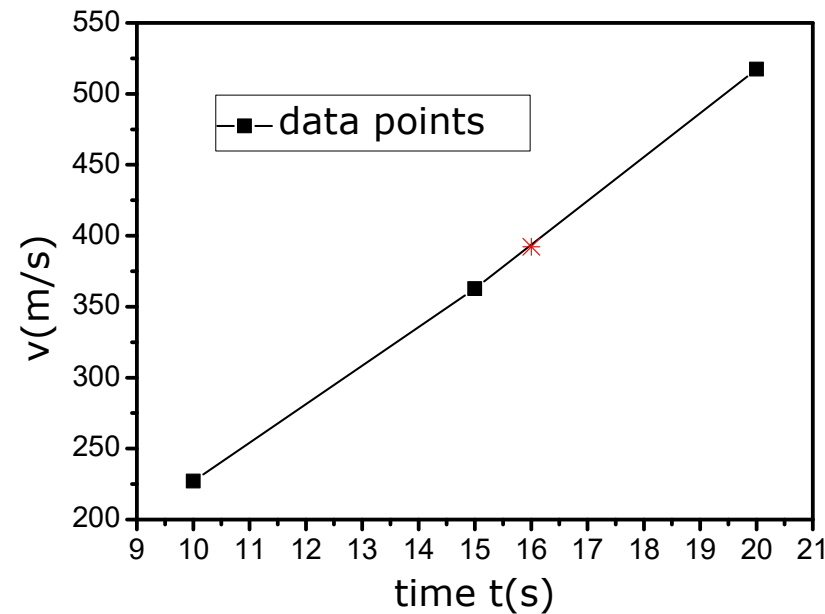
Quadratic interpolation

for $t=16s$:

$$\begin{aligned}v(16) &= \frac{(16-15)(16-20)}{(10-15)(10-20)}(227.04) + \frac{(16-10)(16-20)}{(15-10)(15-20)}(362.78) \\&+ \frac{(16-10)(16-15)}{(20-10)(20-15)}(517.35) = \\&= (-0.08)(227.04) + (0.96)(362.78) + (0.12)(517.35) \\&= 392.19 \text{ m/s}\end{aligned}$$

Homework: please compare obtained result with the direct method and the Newton method of interpolation

Quadratic interpolation



The relative error in comparison with the previous interpolation

$$\begin{aligned}
 |\epsilon_a| &= \left| \frac{392.19 - 393.70}{392.19} \right| \times 100 \\
 &= 0.38502\%
 \end{aligned}$$

Cubic interpolation

Homework

Find the equation expressing speed and calculate v (16s) using Lagrange cubic interpolation

Data

$$t_0 = 10, \quad v(t_0) = 227.04$$

$$t_1 = 15, \quad v(t_1) = 362.78$$

$$t_2 = 20, \quad v(t_2) = 517.35$$

$$t_3 = 22.5, \quad v(t_3) = 602.97$$

Find the distance covered from $t = 11$ s to $t = 16$ s.

Find the acceleration at $t = 16$ s

Compare the results with those obtained on the basis of the direct interpolation method and Newton method.

Comparison

Polynomial order	1	2	3
$v(t=16)$ m/s	393.69	392.19	392.06
The relative error	-----	0.38502 %	0.033427 %

Interpolation by Lagrange polynomial - example

Suppose we are given points: 0, 1, 3, 6. Find the Lagrange interpolation polynomial

$$f(x) = 2 \cdot \sin\left(\frac{\pi}{6} \cdot x\right)$$

Solution:

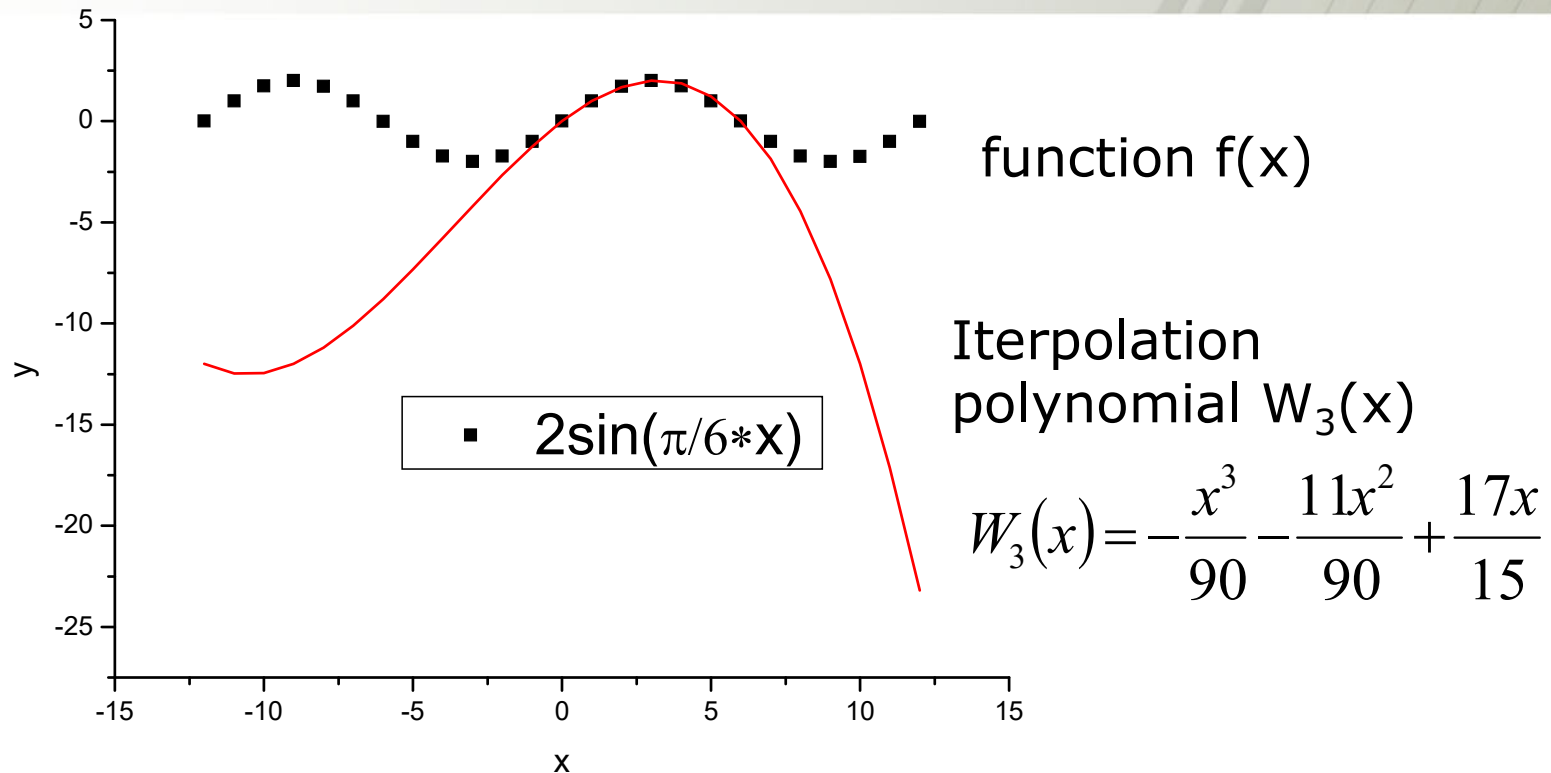
The values of the function $f(x)$ at the interpolation nodes are as follows:

$$y_0 = f(0) = 0, \quad y_1 = f(1) = 1, \quad y_2 = f(3) = 2, \quad y_3 = f(6) = 0.$$

It can be shown that the Lagrange polynomial interpolation is:

$$W_3(x) = -\frac{x^3}{90} - \frac{11x^2}{90} + \frac{17x}{15}$$

Interpolation by Lagrange polynomial - example



The interpolation polynomial approximates the function $f(x)$ only in the interval $[0, 6]$.

The smaller distance between the nodes, the better the approximation.

Spline Method of Interpolation

Motivation

Disadvantages of polynomial interpolation:

Worse results of interpolation when the number of nodes is increased.

Example: $f(x) = |x|$

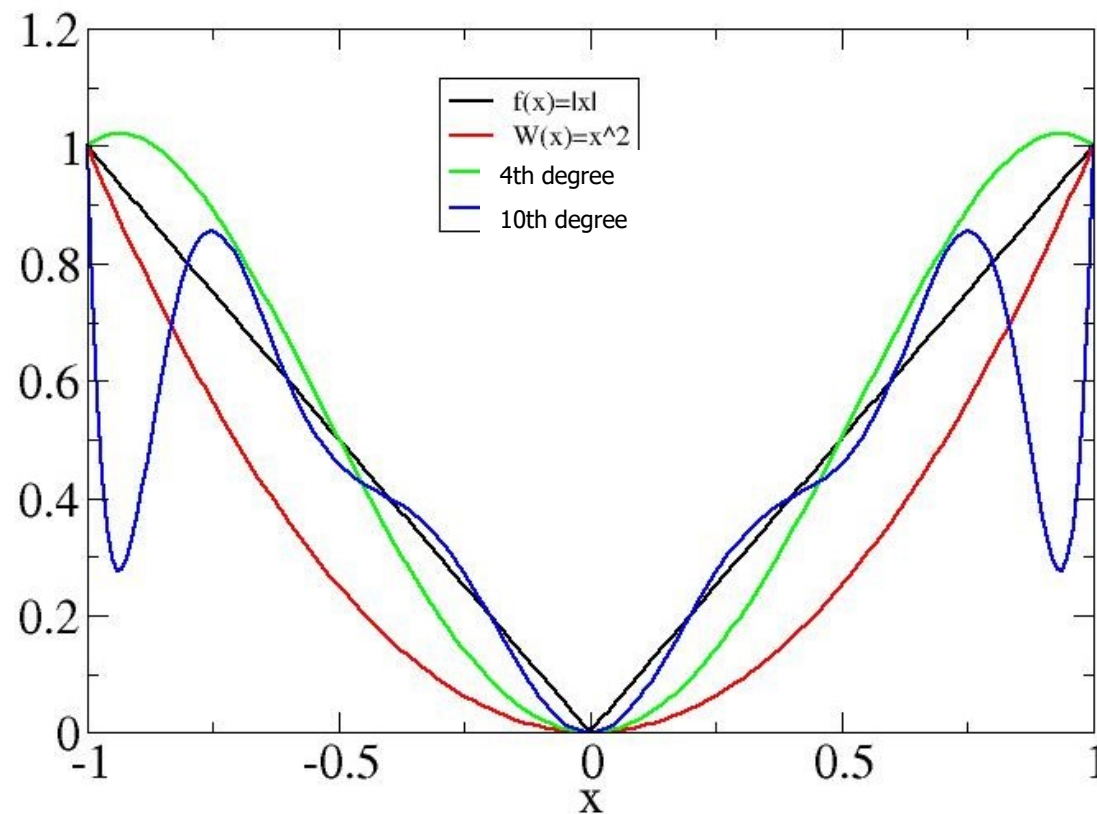
Runge's phenomenon (example of ill- conditioned task):

Polynomial interpolation of high degree at constant node distances leads to a serious deviation from the interpolated function especially at the ends of the interval. Interpolation of the central parts of the range is, however very good and useful.

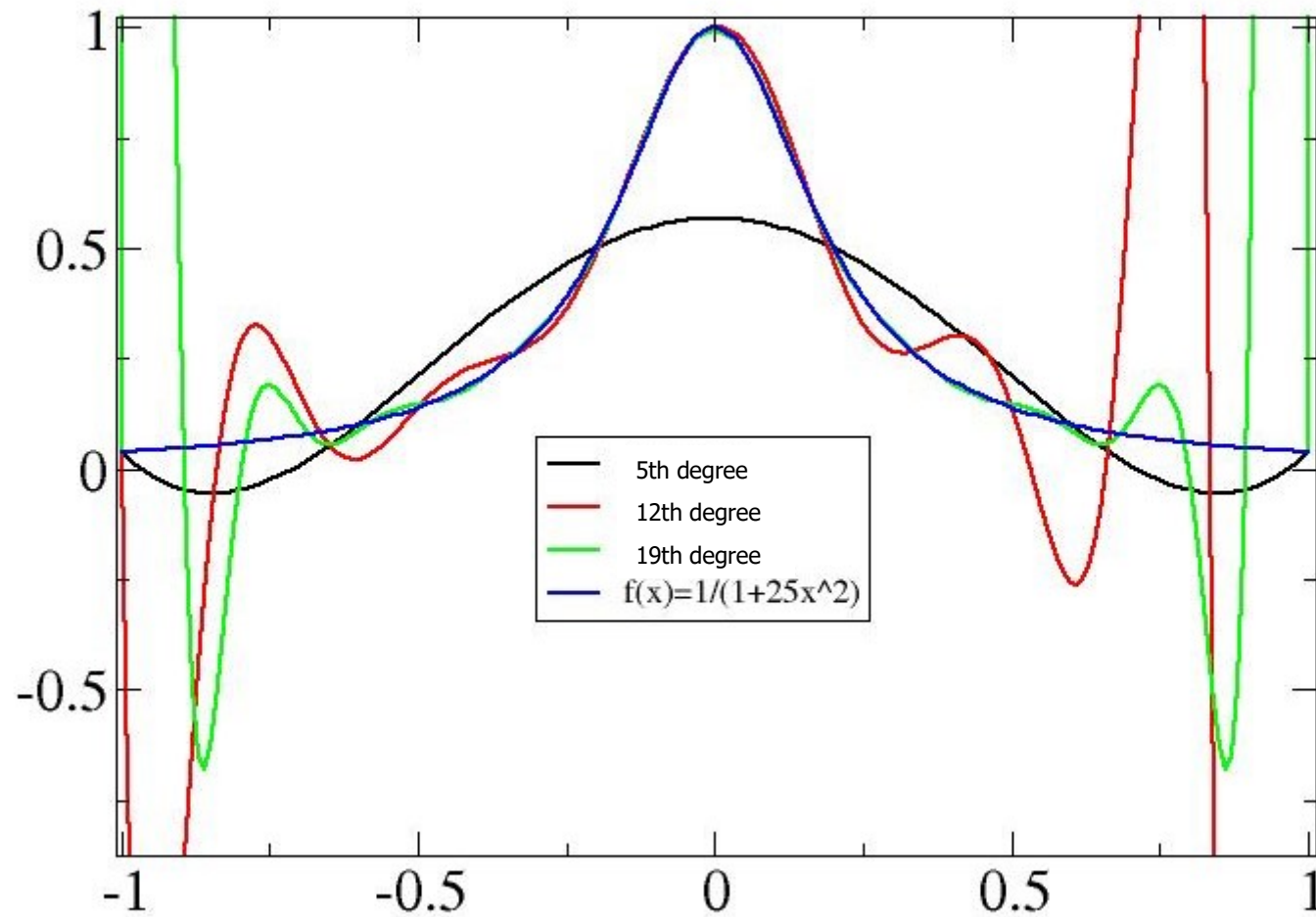
Example: $f(x) = \frac{1}{1+25x^2}$

Polynomial interpolation - example

$$f(x) = |x|$$



Runge's phenomenon

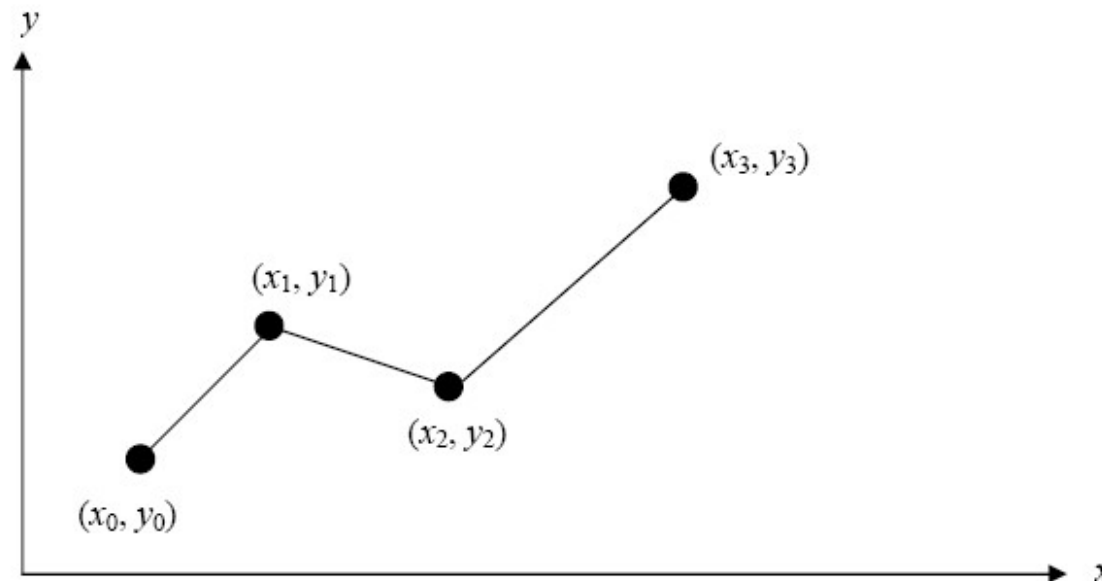


(Linear) Spline Method of Interpolation

With data points:

$$(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$$

we draw straight lines between points.



(Linear) Spline Method of Interpolation

$$f(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0} (x - x_0) \quad x_0 \leq x \leq x_1$$

$$f(x) = f(x_1) + \underbrace{\frac{f(x_2) - f(x_1)}{x_2 - x_1}}_{\text{slope between nodes}} (x - x_1) \quad x_1 \leq x \leq x_2$$

▪

slope between
nodes

▪

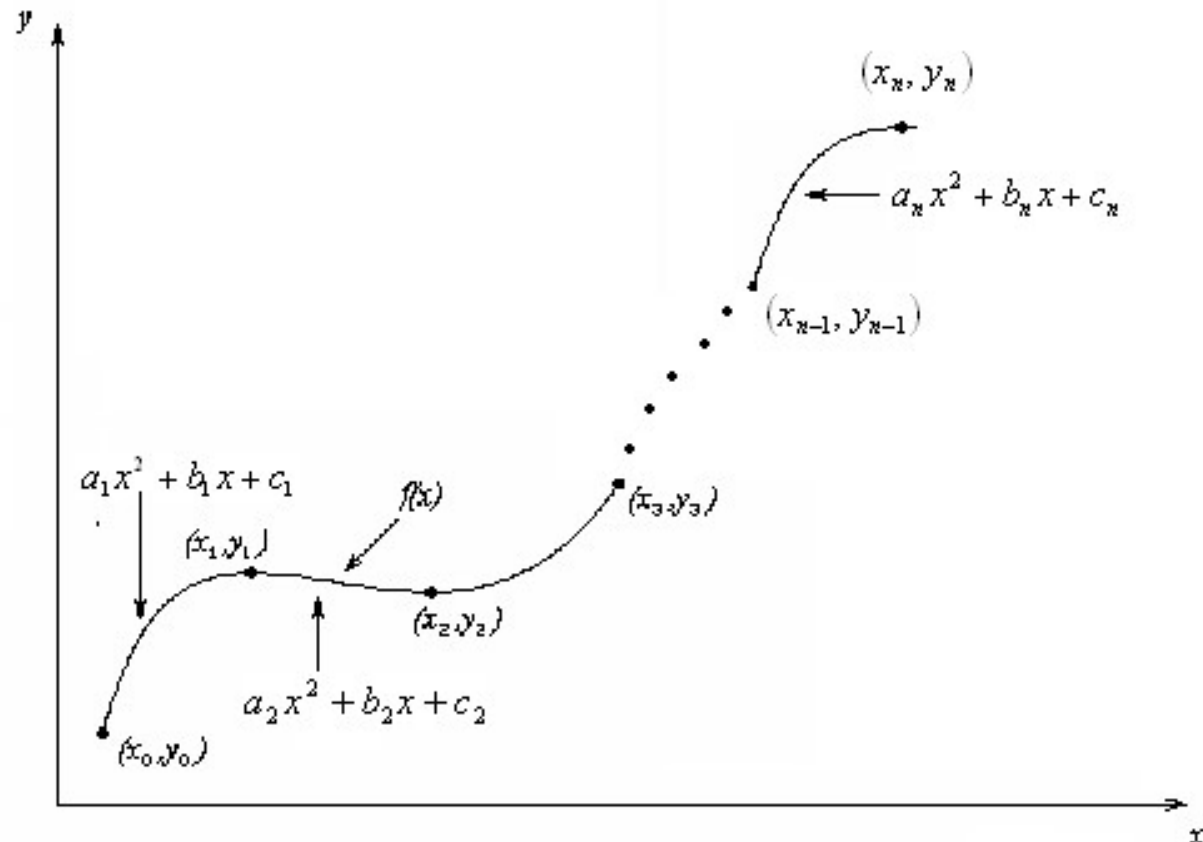
▪

$$f(x) = f(x_{n-1}) + \underbrace{\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}}_{\text{slope between nodes}} (x - x_{n-1}) \quad x_{n-1} \leq x \leq x_n$$

(Quadratic) Spline Method of Interpolation

With data points: $(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$

we write a variety of quadratic functions between each pair of points.



(Quadratic) Spline Method of Interpolation

$$f(x) = a_1x^2 + b_1x + c_1 \quad x_0 \leq x \leq x_1$$

$$f(x) = a_2x^2 + b_2x + c_2 \quad x_1 \leq x \leq x_2$$

▪

▪

▪

$$f(x) = a_nx^2 + b_nx + c_n \quad x_{n-1} \leq x \leq x_n$$

Find the coefficients a_i, b_i, c_i $i = 1, 2, \dots, n$

We have $3n$ unknown values so we need $3n$ equations

(Quadratic) Spline Method of Interpolation

Each parabola passes through two points, so we have $2n$ equations

$$f(x_0) = a_1 x_0^2 + b_1 x_0 + c_1$$

$$f(x_1) = a_1 x_1^2 + b_1 x_1 + c_1$$

⋮

$$f(x_{i-1}) = a_i x_{i-1}^2 + b_i x_{i-1} + c_i$$

$$f(x_i) = a_i x_i^2 + b_i x_i + c_i$$

⋮

$$f(x_{n-1}) = a_n x_{n-1}^2 + b_n x_{n-1} + c_n$$

$$f(x_n) = a_n x_n^2 + b_n x_n + c_n$$

(Quadratic) Spline Method of Interpolation

Additional conditions imposing continuity of the first derivatives in $n-1$ internal nodes can be obtained :

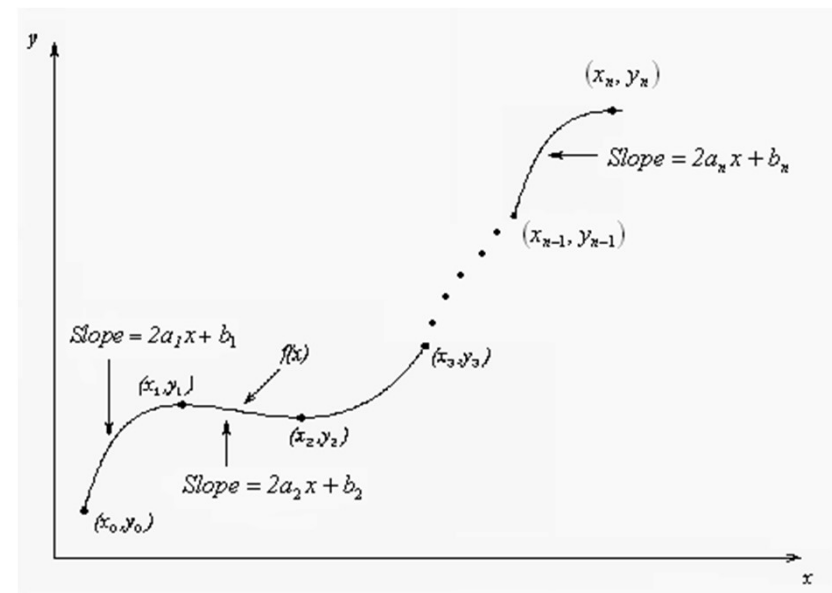
$$\text{for } f(x) \begin{cases} a_1x^2 + b_1x + c_1 \\ a_2x^2 + b_2x + c_2 \end{cases} \begin{matrix} \Rightarrow \\ \Rightarrow \end{matrix} f'(x) \begin{cases} 2a_1x + b_1 \\ 2a_2x + b_2 \end{cases}$$

SO:

$$2a_1x_1 + b_1 = 2a_2x_1 + b_2$$

▪
▪
▪

$$2a_{n-1}x_{n-1} + b_{n-1} = 2a_nx_{n-1} + b_n$$



(Quadratic) Spline Method of Interpolation

As a results, we get $n-1$ equations: $2a_1x_1 + b_1 - 2a_2x_1 - b_2 = 0$

$$2a_2x_2 + b_2 - 2a_3x_2 - b_3 = 0$$

⋮

$$2a_ix_i + b_i - 2a_{i+1}x_i - b_{i+1} = 0$$

⋮

$$2a_{n-1}x_{n-1} + b_{n-1} - 2a_nx_{n-1} - b_n = 0$$

The total number of equations is $2n+(n-1)=3n-1$

The one needed equation may take a form of $a_1 = 0$

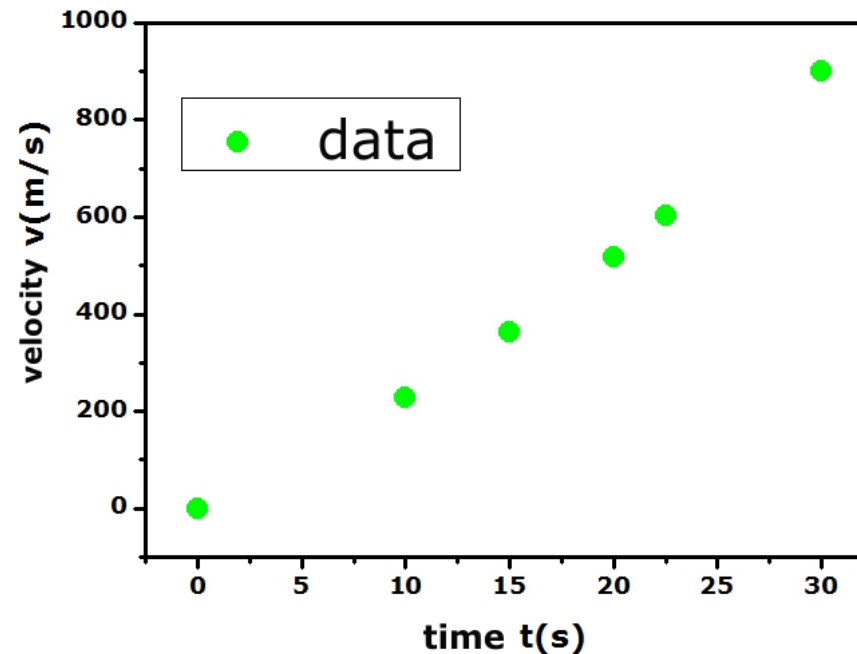
The first spline function is linear.

Example



Table 1 Velocity v as a function of time t

$t(s)$	$v(m/s)$
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67



Find speed at $t = 16$ s using the quadratic spline interpolation

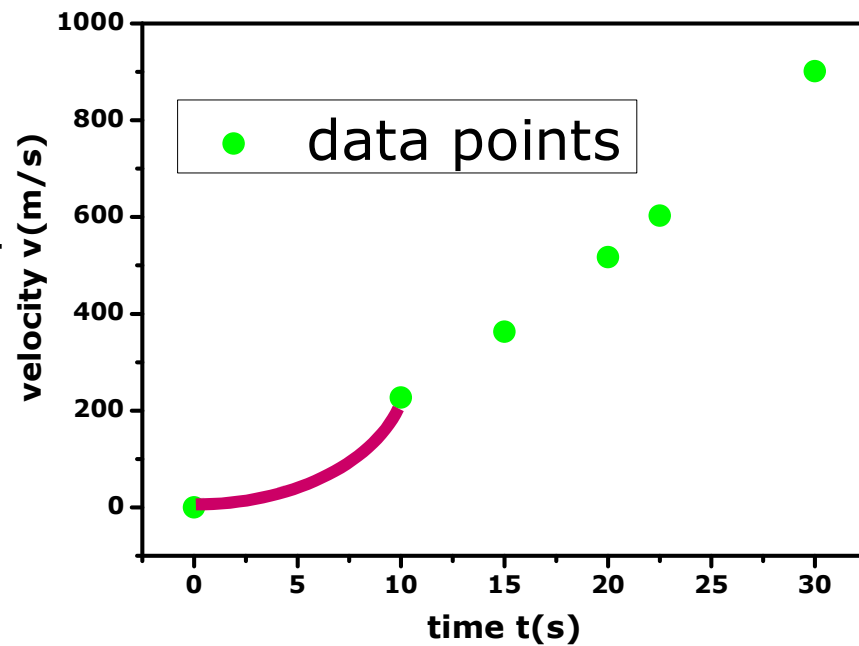
Solution

$$\begin{aligned}v(t) &= a_1 t^2 + b_1 t + c_1, & 0 \leq t \leq 10 \\&= a_2 t^2 + b_2 t + c_2, & 10 \leq t \leq 15 \\&= a_3 t^2 + b_3 t + c_3, & 15 \leq t \leq 20 \\&= a_4 t^2 + b_4 t + c_4, & 20 \leq t \leq 22.5 \\&= a_5 t^2 + b_5 t + c_5, & 22.5 \leq t \leq 30\end{aligned}$$

Each spline goes through two adjacent data points

$$v(t) = a_1 t^2 + b_1 t + c_1, \quad 0 \leq t \leq 10$$

$$\begin{cases} a_1(0)^2 + b_1(0) + c_1 = 0 \\ a_1(10)^2 + b_1(10) + c_1 = 227.04 \end{cases}$$



More equations

t(s)	v(m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

There are 10 equations,
15 coefficients to calculate

$$a_2(10)^2 + b_2(10) + c_2 = 227.04$$

$$a_2(15)^2 + b_2(15) + c_2 = 362.78$$

$$a_3(15)^2 + b_3(15) + c_3 = 362.78$$

$$a_3(20)^2 + b_3(20) + c_3 = 517.35$$

$$a_4(20)^2 + b_4(20) + c_4 = 517.35$$

$$a_4(22.5)^2 + b_4(22.5) + c_4 = 602.97$$

$$a_5(22.5)^2 + b_5(22.5) + c_5 = 602.97$$

$$a_5(30)^2 + b_5(30) + c_5 = 901.67$$

The continuity of derivatives

$$v(t) = a_1 t^2 + b_1 t + c_1, \quad 0 \leq t \leq 10$$

$$= a_2 t^2 + b_2 t + c_2, \quad 10 \leq t \leq 15$$

$$\left. \frac{d}{dt} (a_1 t^2 + b_1 t + c_1) \right|_{t=10} = \left. \frac{d}{dt} (a_2 t^2 + b_2 t + c_2) \right|_{t=10}$$

$$(2a_1 t + b_1) \Big|_{t=10} = (2a_2 t + b_2) \Big|_{t=10}$$

$$2a_1(10) + b_1 = 2a_2(10) + b_2$$

$$20a_1 + b_1 - 20a_2 - b_2 = 0$$

The continuity of derivatives – cont.

$$\text{for } t=10\text{s} \quad 2a_1(10) + b_1 - 2a_2(10) - b_2 = 0$$

$$\text{for } t=15\text{s} \quad 2a_2(15) + b_2 - 2a_3(15) - b_3 = 0$$

$$\text{for } t=20\text{s} \quad 2a_3(20) + b_3 - 2a_4(20) - b_4 = 0$$

$$\text{for } t=22.5\text{s} \quad 2a_4(22.5) + b_4 - 2a_5(22.5) - b_5 = 0$$

4 additional equations

The last equation $a_1 = 0$

The final matrix of 15 equations and 15 unknowns

$$\begin{bmatrix}
 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 100 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 100 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 225 & 15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 225 & 15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 400 & 20 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 400 & 20 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 506.25 & 22.5 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 506.25 & 22.5 & 1 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 900 & 30 & 1 \\
 20 & 1 & 0 & -20 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 30 & 1 & 0 & -30 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 40 & 1 & 0 & -40 & -1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 45 & 1 & 0 & -45 & -1 & 0 \\
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
 \end{bmatrix}
 \begin{bmatrix}
 a_1 \\
 b_1 \\
 c_1 \\
 a_2 \\
 b_2 \\
 c_2 \\
 a_3 \\
 b_3 \\
 c_3 \\
 a_4 \\
 b_4 \\
 c_4 \\
 a_5 \\
 b_5 \\
 c_5
 \end{bmatrix}
 =
 \begin{bmatrix}
 0 \\
 227.04 \\
 227.04 \\
 362.78 \\
 362.78 \\
 517.35 \\
 517.35 \\
 602.97 \\
 602.97 \\
 901.67 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0
 \end{bmatrix}$$

b_i 

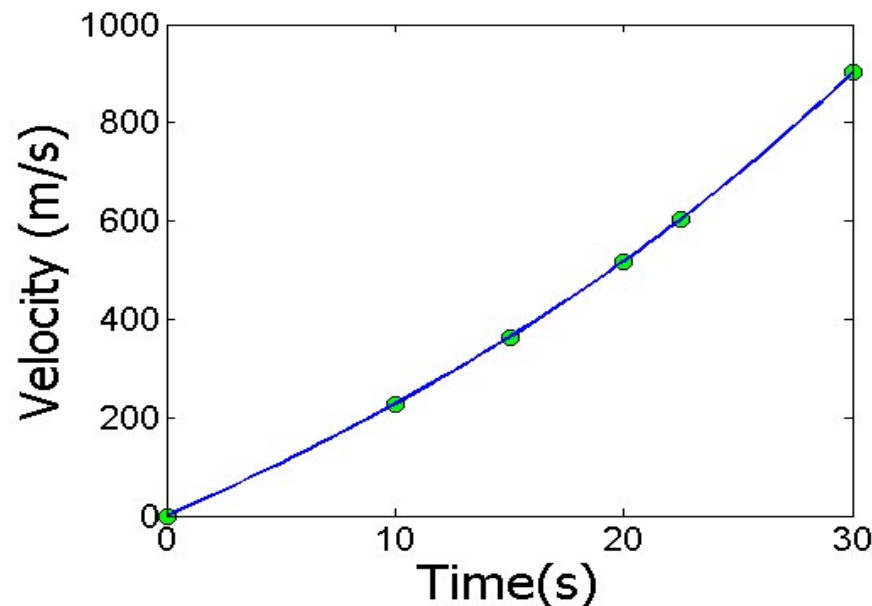
Solution

i	a_i	b_i	c_i
1	0	22.704	0
2	0.8888	4.928	88.88
3	-0.1356	35.66	-141.61
4	1.6048	-33.956	554.55
5	0.20889	28.86	-152.13

Make sure that these values are correct

Solution

$$\begin{aligned}
 v(t) &= 22.704t, & 0 \leq t \leq 10 \\
 &= 0.8888t^2 + 4.928t + 88.88, & 10 \leq t \leq 15 \\
 &= -0.1356t^2 + 35.66t - 141.61, & 15 \leq t \leq 20 \\
 &= 1.6048t^2 - 33.956t + 554.55, & 20 \leq t \leq 22.5 \\
 &= 0.20889t^2 + 28.86t - 152.13, & 22.5 \leq t \leq 30
 \end{aligned}$$



Velocity at a specific moment

a) Velocity at $t=16s$

$$\begin{aligned}v(t) &= 22.704t, & 0 \leq t \leq 10 \\&= 0.8888t^2 + 4.928t + 88.88, & 10 \leq t \leq 15 \\&= -0.1356t^2 + 35.66t - 141.61, & 15 \leq t \leq 20 \\&= 1.6048t^2 - 33.956t + 554.55, & 20 \leq t \leq 22.5 \\&= 0.20889t^2 + 28.86t - 152.13, & 22.5 \leq t \leq 30\end{aligned}$$

$$\begin{aligned}v(16) &= -0.1356(16)^2 + 35.66(16) - 141.61 \\&= 394.24 \text{ m/s}\end{aligned}$$

As a homework, please compare the calculated value of the velocity with the value obtained by polynomial interpolation

Acceleration at a specific moment

b) Acceleration at $t=16s$

$$v(t) = 22.704t, \quad 0 \leq t \leq 10$$

$$= 0.8888 t^2 + 4.928 t + 88.88, \quad 10 \leq t \leq 15$$

$$= -0.1356 t^2 + 35.66t - 141.61, \quad 15 \leq t \leq 20$$

$$= 1.6048 t^2 - 33.956t + 554.55, \quad 20 \leq t \leq 22.5$$

$$= 0.20889 t^2 + 28.86t - 152.13, \quad 22.5 \leq t \leq 30$$

$$a(16) = \frac{d}{dt} v(t) \Big|_{t=16}$$

Acceleration at a specific moment

The quadratic spline function at the point $t = 16\text{s}$ is given as:

$$v(t) = -0.1356t^2 + 35.66t - 141.61, \quad 15 \leq t \leq 20$$

$$\begin{aligned} a(t) &= \frac{d}{dt}(-0.1356t^2 + 35.66t - 141.61) \\ &= -0.2712t + 35.66, \end{aligned}$$

$$a(16) = -0.2712(16) + 35.66 = 31.321\text{m/s}^2$$

As a homework, please compare the calculated value of the acceleration with the value obtained by polynomial interpolation

Calculation of a distance covered

c) Find the distance covered by the rocket between $t=11$ s to $t=16$ s.

$$\begin{aligned}v(t) &= 22.704t, & 0 \leq t \leq 10 \\&= 0.8888t^2 + 4.928t + 88.88, & 10 \leq t \leq 15 \\&= -0.1356t^2 + 35.66t - 141.61, & 15 \leq t \leq 20 \\&= 1.6048t^2 - 33.956t + 554.55, & 20 \leq t \leq 22.5 \\&= 0.20889t^2 + 28.86t - 152.13, & 22.5 \leq t \leq 30\end{aligned}$$

$$S(16) - S(11) = \int_{11}^{16} v(t) dt$$

Calculation of a distance covered

$$v(t) = 0.8888t^2 + 4.928t + 88.88, \quad 10 \leq t \leq 15$$

$$= -0.1356t^2 + 35.66t - 141.61, \quad 15 \leq t \leq 20$$

$$\begin{aligned} S(16) - S(11) &= \int_{11}^{16} v(t) dt = \int_{11}^{15} v(t) dt + \int_{15}^{16} v(t) dt \\ &= \int_{11}^{15} (0.8888t^2 + 4.928t + 88.88) dt \\ &\quad + \int_{15}^{16} (-0.1356t^2 + 35.66t - 141.61) dt = 1595.9 \text{ m} \end{aligned}$$

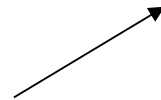
As a homework, please compare the calculated value of the distance with the value obtained by polynomial interpolation

Error of interpolation

$$|f(x) - W_n(x)| \leq \frac{\sup_{x \in \langle a, b \rangle} |f^{(n+1)}(x)|}{(n+1)!} \cdot \left| \prod_{i=0}^n (x - x_i) \right|$$

Assuming that:

$$M_{n+1} = \sup_{x \in \langle a, b \rangle} |f^{(n+1)}(x)|$$



The upper limit of the module $(n+1)^{\text{th}}$ derivative of $f(x)$ in $\langle a, b \rangle$

$$\omega_n(x) = (x - x_0)(x - x_1) \dots (x - x_n)$$

Error of interpolation

$$|f(x) - W_n(x)| \leq \frac{M_{n+1}}{(n+1)!} \cdot |\omega_n(x)|$$

Example:

Assess with what accuracy you can calculate the value of $\ln 100.5$ using Lagrange interpolation formula, if the given values are: $\ln 100$ $\ln 101$ $\ln 102$ $\ln 103$

$$f(x) = \ln(x), \quad n = 3, \quad a = 100, \quad b = 103, \quad f^{(4)}(x) = -\frac{6}{x^4}$$

$$M_4 = \sup_{x \in \langle 100, 103 \rangle} |f^{(4)}(x)| = \frac{6}{100^4}$$

$$|\ln 100,5 - W(100,5)| \leq \frac{6}{100^4 4!} \cdot 0,5 \cdot 0,5 \cdot 1,5 \cdot 2,5 \approx 2,344 \cdot 10^{-9}$$

The optimum choice of interpolation nodes

$$|f(x) - W_n(x)| \leq \frac{M_{n+1}}{(n+1)!} \cdot |\omega_n(x)|$$

The value of the error depends on the choice of the interpolation nodes - ω_n . We cannot affect the M_{n+1}

How to choose the interpolation nodes x_i to achieve the lowest value of:

$$\sup_{x \in \langle a, b \rangle} |\omega_n(x)|$$

This issue was formulated by Russian mathematician P.L. Chebyshev as a problem of finding the best algebraic polynomial approximating zero in a given interval.

Chebyshev polynomials

Chebyshev polynomials (of the first type):

$$T_n(x) = \cos(n \operatorname{arc} \cos x)$$

It can be shown that the polynomial $T_n(x)$ is identical with a certain algebraic polynomial limited to the interval $\langle -1, 1 \rangle$.

$$T_0(x) = 1$$

$$T_1(x) = \cos(\operatorname{arc} \cos x) = x$$

$$T_2(x) = \cos(2 \operatorname{arc} \cos x) = 2x^2 - 1$$

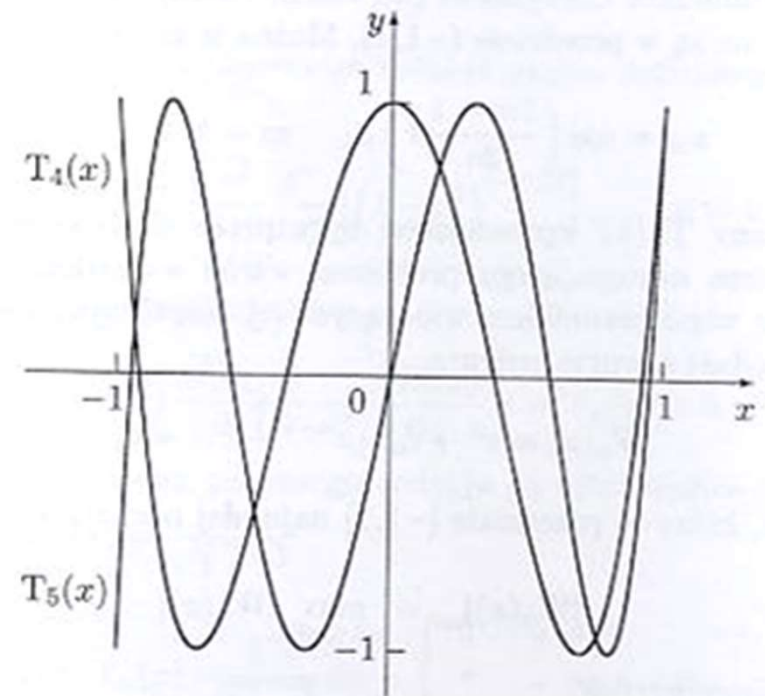
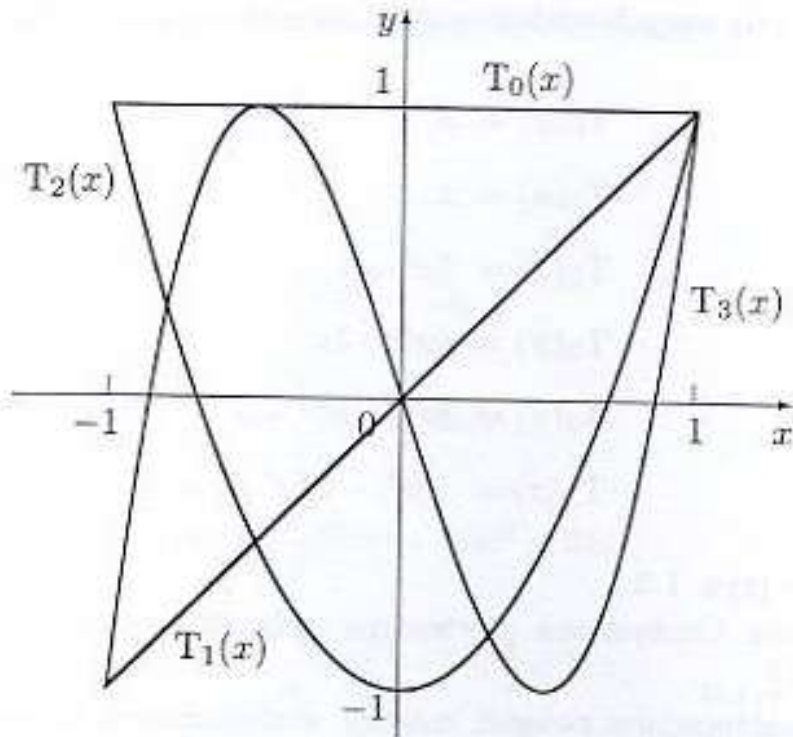
$$T_3(x) = \cos(3 \operatorname{arc} \cos x) = 4x^3 - 3x$$

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$$

Chebyshev polynomials of the first type

$$T_0(x) = 1$$

$$T_1(x) = \cos(\arccos x) = x$$



Chebyshev polynomials

Chebyshev polynomials of the first type are the solution of the differential equation:

$$(1-x^2) \frac{d^2 T_n(x)}{dx^2} - x \frac{dT_n(x)}{dx} + n^2 T_n(x) = 0$$

They are defined by the Rodrigues formula:

$$T_n(x) = (-1)^n \frac{\sqrt{1-x^2}}{(2n-1)!!} \frac{d^n}{dx^n} \left[(1-x^2)^{n-1/2} \right]$$

Chebyshev polynomials of the first type are orthogonal in the interval

$\langle -1, 1 \rangle$ with a weight:

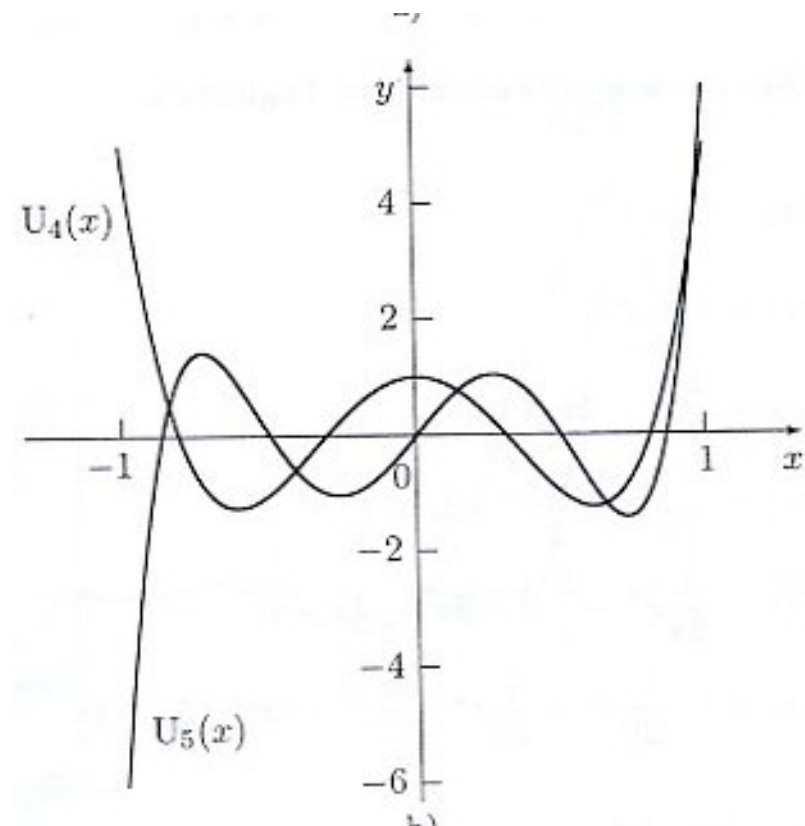
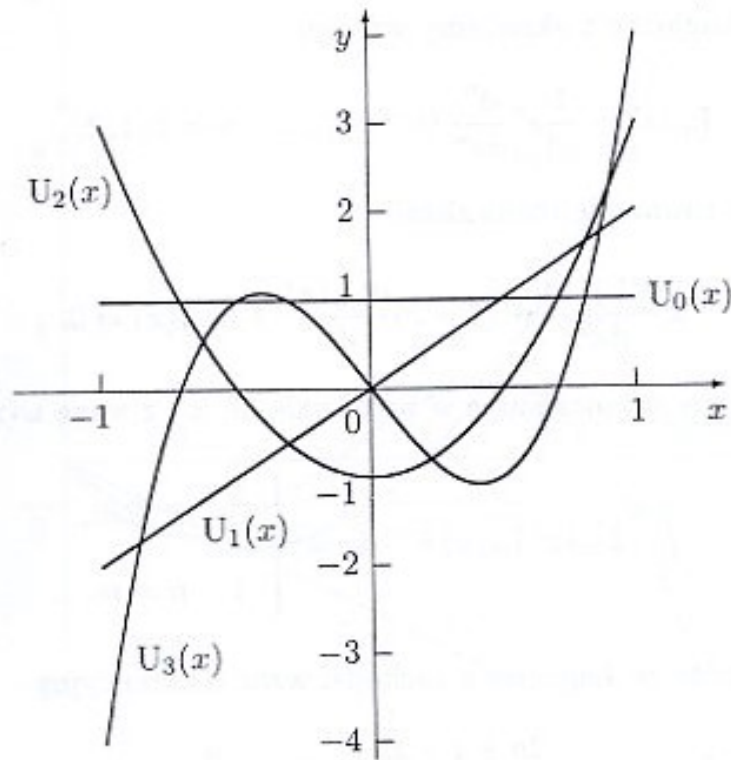
$$w(x) = \frac{1}{\sqrt{1-x^2}}$$

Chebyshev polynomials of the second type

$$U_0(x) = 1$$

$$U_1(x) = 2x$$

$$U_2(x) = 4x^2 - 1$$



Chebyshev polynomials

Chebyshev polynomials of the second type are the solution of the differential equation:

$$(1-x^2)\frac{d^2U_n(x)}{dx^2} - 3x\frac{dU_n(x)}{dx} + n(n+2)U_n(x) = 0$$

They are defined by the Rodrigues formula

$$U_n(x) = (-1)^n \frac{1}{\sqrt{1-x^2}} \frac{n+1}{(2n+1)!!} \frac{d^n}{dx^n} \left[(1-x^2)^{n+1/2} \right]$$

Chebyshev polynomials of the second type are orthogonal in the interval $\langle -1, 1 \rangle$ with a weight :

$$w(x) = \sqrt{1-x^2}$$

The optimal choice of interpolation nodes

Every Chebyshev polynomial of a degree n has roots at:

$$x_m = \cos\left(\frac{2m+1}{2n}\pi\right), \quad m = 0, 1, 2, \dots, n-1$$

between -1 i $+1$

Coefficient of the highest power of $T_n(x)$ is equal to 2^{n-1} .

We are looking for a polynomial, which has the highest power factor of unity

$$T_{n+1}^*(x) = \frac{1}{2^n} T_{n+1}(x) = (x - x_0)(x - x_1) \dots (x - x_n)$$

where $x_m (m=0, 1, 2, \dots, n)$ are the roots of a polynomial T_{n+1}

The optimal choice of interpolation nodes

Expression: $\sup_{x \in \langle a, b \rangle} |\omega_n(x)|$

in the interval $\langle -1, 1 \rangle$ has the lowest value for the polynomial :

$$\omega_n(x) = \frac{1}{2^n} T_{n+1}(x) = (x - x_0)(x - x_1) \dots (x - x_n)$$

in this case: $\sup_{x \in \langle -1, 1 \rangle} |\omega_n(x)| = \frac{1}{2^n}$

When, in the interval $\langle -1, 1 \rangle$ we assume Chebyshev polynomial zeros as the interpolation nodes , then:

$$|f(x) - W_n(x)| \leq \frac{M_{n+1}}{2^n (n+1)!}$$

The optimal choice of interpolation nodes

At any range $\langle a, b \rangle$ estimated error is:

$$|f(x) - W_n(x)| \leq \frac{M_{n+1}}{(n+1)!} \frac{(b-a)^{n+1}}{2^{2n+1}}$$

the selection of nodes:

$$x_m = \frac{1}{2} \left[(b-a) \cos \frac{2m+1}{2n+2} \pi + (b+a) \right], \quad m = 0, 1, 2, \dots, n$$

The new nodes x_m are not evenly spaced, but are concentrated at the ends of the interval.

Simple linear transformations convert x from the interval $\langle a, b \rangle$ to z in the interval $\langle -1, 1 \rangle$

$$x = \frac{1}{2} [(b-a)z + (b+a)]$$

$$z = \frac{1}{b-a} (2x - b - a)$$

Interpolation - summary

Conclusions:

1. When calculating the values of the polynomial interpolation in one or more points the problem of selecting a interpolation formula is not important.
2. The selected formula and distribution of nodes affect only the calculation error.
3. The number of multiplications and division mainly determines the total time needed for calculations

for the Lagrange polynomial n^2+4n+2

For the Newton polynomial $\frac{1}{2} n^2+\frac{3}{2} n$