

Numerical Methods

Lecture 4.

Numerical integration

Prof. dr hab. inż. Katarzyna Zakrzewska

Outline

- **Trapezoidal rule**
- **Multi-segment trapezoidal rule**
- **Richardson extrapolation**
- **Romberg's method**
- **Simpson's rule**
- **Gaussian method**

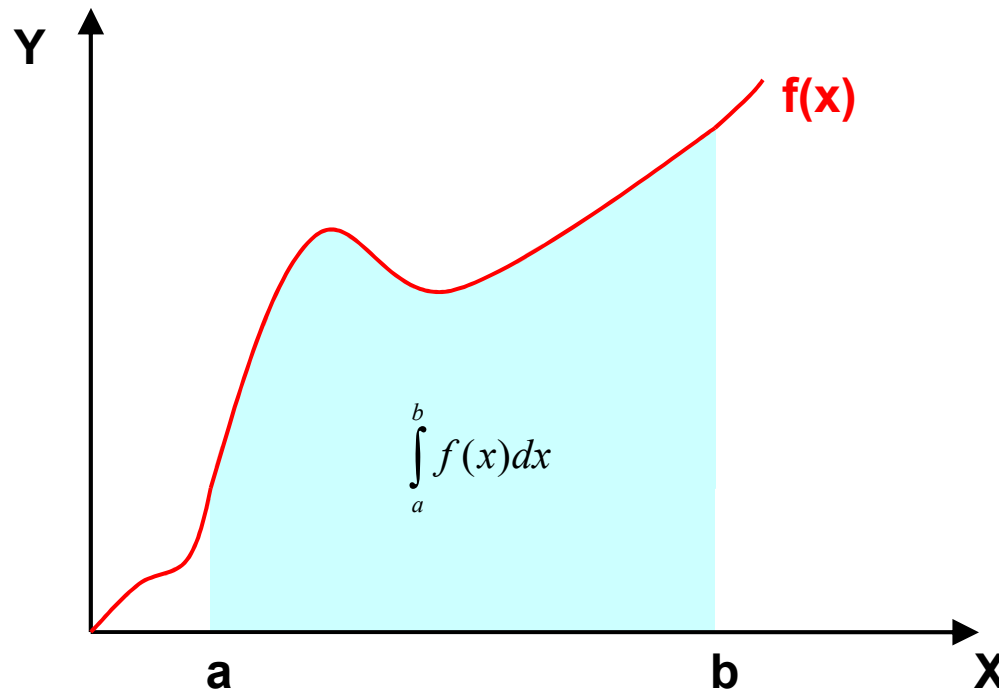
Numerical integration - idea

$$I = \int_a^b f(x) dx$$

The integral can be approximated by the sum of the

$$S = \sum_{i=1}^n f(c_i) \Delta x_i$$

$$x_i \leq c_i \leq x_{i+1}$$



Newton–Cotes methods

Newton – Cotes integration belongs to a class of methods with fixed nodes: function $f(x)$ is interpolated by a polynomial (e.g. Lagrange polynomial)

$$f(x) \approx f_n(x)$$

where:

$$f_n(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1} + a_nx^n$$

Then, the integral of $f(x)$ can be approximated as an integral of the interpolated function $f_n(x)$

$$I = \int_a^b f(x)dx \approx \int_a^b f_n(x)dx$$

Trapezoidal rule

The trapezoidal rule
assumes: $n = 1$, thus:

$$f_1(x) = a_0 + a_1x$$

$$\begin{aligned}
 I = \int_a^b f(x)dx &\approx \int_a^b f_1(x)dx = \int_a^b (a_0 + a_1x)dx \\
 &= a_0(b-a) + a_1 \frac{b^2 - a^2}{2}
 \end{aligned}$$

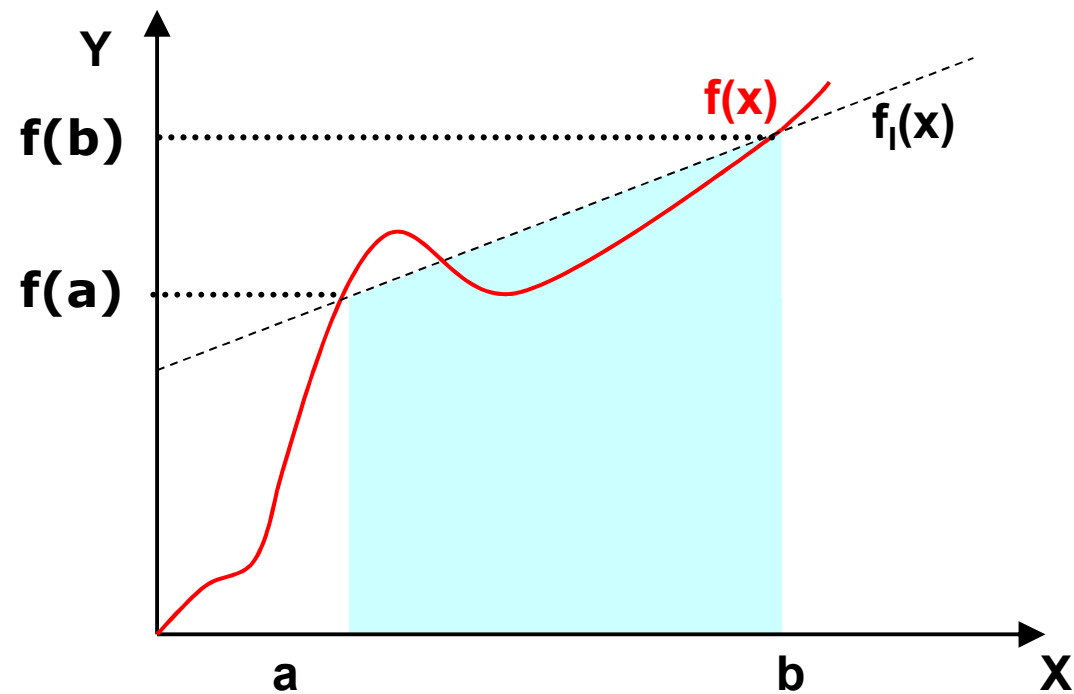
We are looking for a_0 and a_1 ?

Now if one chooses, $(a, f(a))$ i $(b, f(b))$ as the two points to approximate $f(x)$ by a straight line from a to b , then:

$$\left\{ \begin{array}{ll} f(a) = f_1(a) = a_0 + a_1a & \Rightarrow a_0 = \frac{f(a)b - f(b)a}{b-a} \\ f(b) = f_1(b) = a_0 + a_1b & \Rightarrow a_1 = \frac{f(b) - f(a)}{b-a} \end{array} \right.$$

Trapezoidal rule

$$\int_a^b f(x)dx = (b-a) \left[\frac{f(a) + f(b)}{2} \right] \quad \int_a^b f(x)dx \approx \text{area of trapezoid}$$



Trapezoidal rule

Example 1:

The vertical distance covered by a rocket from $t_1=8$ s to $t_2=30$ s is given by:

$$v(t) = 2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t$$

- a) Use the single segment trapezoidal rule to find the distance covered from $t_1=8$ s to $t_2=30$ s
- b) Find the true relative error.

Trapezoidal rule

$$a) \quad I \approx (b-a) \left[\frac{f(a) + f(b)}{2} \right] \quad a = 8 \text{ s} \quad b = 30 \text{ s}$$

$$f(t) = 2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t$$

$$f(8) = 2000 \ln \left[\frac{140000}{140000 - 2100(8)} \right] - 9.8(8) = 177.27 \text{ m/s}$$

$$f(30) = 2000 \ln \left[\frac{140000}{140000 - 2100(30)} \right] - 9.8(30) = 901.67 \text{ m/s}$$

$$I = (30 - 8) \left[\frac{177.27 + 901.67}{2} \right] = 11868 \text{ m}$$

Trapezoidal rule

b) The true value

$$\Delta x = \int_8^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt = 11061 \text{ m}$$

The relative error:

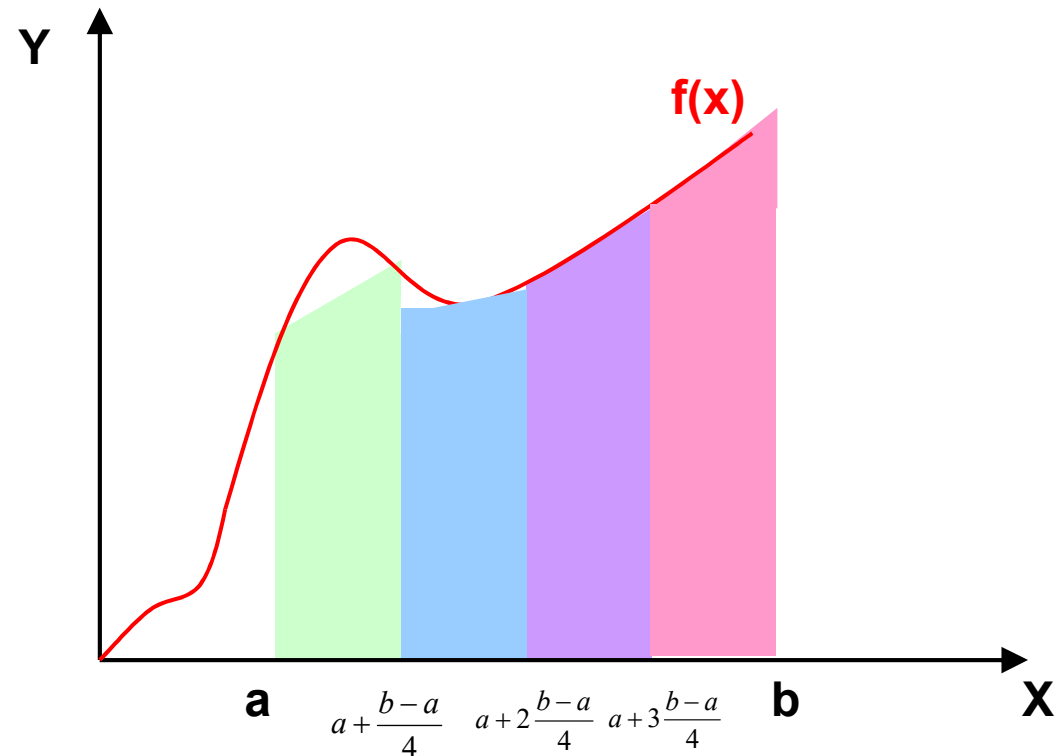
$$|\epsilon_t| = \left| \frac{11061 - 11868}{11061} \right| \times 100 = 7.2959 \%$$

Multi-segment trapezoidal rule

The true error using a single segment trapezoidal rule was large. We can divide the interval into n subsections (of equal length - h) and apply the trapezoidal rule over each segment:

$$h = \frac{b-a}{n}$$

for $n=4$



$$I = \int_a^b f(x)dx = \int_a^{a+h} f(x)dx + \int_{a+h}^{a+2h} f(x)dx + \int_{a+2h}^{a+3h} f(x)dx + \int_{a+3h}^{a+4h} f(x)dx$$

Multi-segment trapezoidal rule

$$\int_a^b f(x)dx = \int_a^{a+h} f(x)dx + \int_{a+h}^{a+2h} f(x)dx + \dots + \int_{a+(n-2)h}^{a+(n-1)h} f(x)dx + \int_{a+(n-1)h}^b f(x)dx$$

$$= h \left[\frac{f(a) + f(a+h)}{2} \right] + h \left[\frac{f(a+h) + f(a+2h)}{2} \right] + \dots$$

$$\dots + [b - (a + (n-1)h)] \left[\frac{f(a + (n-1)h) + f(b)}{2} \right]$$

$$\int_a^b f(x)dx = \frac{b-a}{2n} \left[f(a) + 2 \left\{ \sum_{i=1}^{n-1} f(a+ih) \right\} + f(b) \right]$$

Multi-segment trapezoidal rule

Example 2:

The vertical distance covered by a rocket from $t_1=8$ s to $t_2=30$ s seconds is given by:

$$v(t) = 2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t$$

- a) Use the complex segment trapezoidal rule to find the distance covered from $t_1=8$ s to $t_2=30$ s for $n = 2$
- b) Find the true relative error.

Multi-segment trapezoidal rule

a)
$$I = \frac{b-a}{2n} \left[f(a) + 2 \left\{ \sum_{i=1}^{n-1} f(a+ih) \right\} + f(b) \right]$$

$n = 2 \quad a = 8 \text{ s} \quad b = 30 \text{ s} \quad h = \frac{b-a}{n} = \frac{30-8}{2} = 11 \text{ s}$

$$I = \frac{30-8}{2(2)} \left[f(8) + 2 \left\{ \sum_{i=1}^{2-1} f(a+ih) \right\} + f(30) \right]$$

$$= \frac{22}{4} [f(8) + 2f(19) + f(30)] = \frac{22}{4} [177.27 + 2(484.75) + 901.67]$$

$$= 11266 \text{ m}$$

Multi-segment trapezoidal rule

b) the true value:

$$x = \int_8^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt = 11061 \text{ m}$$

The relative error:

$$|\epsilon_t| = \left| \frac{11061 - 11266}{11061} \right| \times 100 = 1.8534\%$$

Multi-segment trapezoidal rule

n	Δx	E_t	$\epsilon_t \%$	$\epsilon_a \%$
1	11868	-807	7.296	---
2	11266	-205	1.853	5.343
3	11153	-91.4	0.8265	1.019
4	11113	-51.5	0.4655	0.3594
5	11094	-33.0	0.2981	0.1669
6	11084	-22.9	0.2070	0.09082
7	11078	-16.8	0.1521	0.05482
8	11074	-12.9	0.1165	0.03560

Trapezoidal rule errors

The absolute true error for a simple trapezoidal rule

$$E_t = \frac{(b-a)^3}{12} f''(\zeta), \quad a < \zeta < b$$

where ζ is a point in: $[a, b]$

The error in the complex trapezoidal rule is a sum of errors for each segment. The relative error of one segment is given by:

$$\begin{aligned} E_1 &= \frac{[(a+h)-a]^3}{12} f''(\zeta_1), \quad a < \zeta_1 < a+h \\ &= \frac{h^3}{12} f''(\zeta_1) \end{aligned}$$

Trapezoidal rule errors

By analogy:

$$E_i = \frac{[(a + ih) - (a + (i - 1)h)]^3}{12} f''(\zeta_i), \quad a + (i - 1)h < \zeta_i < a + ih$$
$$= \frac{h^3}{12} f''(\zeta_i)$$

for n:

$$E_n = \frac{[b - \{a + (n - 1)h\}]^3}{12} f''(\zeta_n), \quad a + (n - 1)h < \zeta_n < b$$
$$= \frac{h^3}{12} f''(\zeta_n)$$

Trapezoidal rule errors

The total error in the complex trapezoidal rule is a sum of the errors of single segments:

$$E_t = \sum_{i=1}^n E_i = \frac{h^3}{12} \sum_{i=1}^n f''(\zeta_i) = \frac{(b-a)^3}{12n^2} \frac{\sum_{i=1}^n f''(\zeta_i)}{n}$$

Formula:

$$\alpha = \frac{\sum_{i=1}^n f''(\zeta_i)}{n}$$

gives an approximate average value of the second derivative in the range of $a < x < b$

$$E_t \propto \alpha \frac{1}{n^2}$$

Trapezoidal rule errors

Table of results for integral $\int_8^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$

as a function of the number of segments – n. When n increases twice the error decreases four times!

n	Value	E_t	$ \epsilon_t \%$	$ \epsilon_a \%$
2	11266	-205	1.854	5.343
4	11113	-51.5	0.4655	0.3594
8	11074	-12.9	0.1165	0.03560
16	11065	-3.22	0.02913	0.00401

Romberg integration method

Romberg method is an extension of the trapezoidal method and gives a better approximation of the integral by reducing the error (true error)

Richardson extrapolation

The true error obtained when using the multiple segment trapezoidal rule (complex) with n segments to approximate an integral is given by:

$$E_t \cong \frac{C}{n^2}$$

where: C is an approximate constant of proportionality

Since:

$$E_t = TV - I_n$$

True value

approximate value using
 n -segments

Richardson extrapolation

It can be shown, that:

$$\frac{C}{(2n)^2} \cong TV - I_{2n}$$

If the number of segments is doubled from n to $2n$ in the trapezoidal rule:

$$\left\{ \begin{array}{l} \frac{C}{(n)^2} \cong TV - I_n \\ \frac{C}{(2n)^2} \cong TV - I_{2n} \end{array} \right.$$

we get:

$$TV \cong I_{2n} + \frac{I_{2n} - I_n}{3}$$

Richardson extrapolation

Example 3:

The vertical distance covered by a rocket from $t_1=8$ s to $t_2=30$ s seconds is given by:

$$v(t) = 2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t$$

- a) Use Richardson extrapolation rule to find the distance covered for $n = 2$
- b) Find the absolute relative true error

Table of results from a complex trapezoid rules for $n = 8$ segments

n	Δx	E_t	$\epsilon_t /\%$	$\epsilon_a /\%$
1	11868	-807	7.296	---
2	11266	-205	1.853	5.343
3	11153	-91.4	0.8265	1.019
4	11113	-51.5	0.4655	0.3594
5	11094	-33.0	0.2981	0.1669
6	11084	-22.9	0.2070	0.09082
7	11078	-16.8	0.1521	0.05482
8	11074	-12.9	0.1165	0.03560

Richardson extrapolation

a)

$$I_2 = 11266m \qquad I_4 = 11113m$$

$$TV \cong I_{2n} + \frac{I_{2n} - I_n}{3} \quad \text{for } n=2$$

$$\begin{aligned} TV &\cong I_4 + \frac{I_4 - I_2}{3} = 11113 + \frac{11113 - 11266}{3} \\ &= 11062m \end{aligned}$$

Richardson extrapolation

b) The true value:

$$x = \int_8^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt = 11061 \text{ m}$$

The absolute true error:

$$E_t = 11061 - 11062 = -1 \text{ m}$$

Richardson extrapolation

c) The relative true error:

$$|\epsilon_t| = \left| \frac{11061 - 11062}{11061} \right| \times 100 = 0.00904 \%$$

Comparison of the results with the trapezoidal rule

n	Δx (m) Trapezoidal rule	$ \epsilon_t \%$ Trapezoidal rule	Δx (m) Richardson extrapolation	$ \epsilon_t \%$ Richardson extrapolation
1	11868	7.296	--	--
2	11266	1.854	11065	0.03616
4	11113	0.4655	11062	0.009041
8	11074	0.1165	11061	0.0000

Romberg's method

Romberg's method uses the same pattern as Richardson extrapolation. However, Romberg's method uses a recursive algorithm for the extrapolation as follows:

$$TV \cong I_{2n} + \frac{I_{2n} - I_n}{3}$$

which can be expressed as:

$$(I_{2n})_R = I_{2n} + \frac{I_{2n} - I_n}{3} = I_{2n} + \frac{I_{2n} - I_n}{4^{2-1} - 1}$$

The true value TV is replaced by result of Richardson extrapolation

$$(I_{2n})_R$$

Note also that the sign $\cong$ is replaced by the sign $=$

Romberg's method

Estimated true value is given by: $TV \cong (I_{2n})_R + Ch^4$

where: Ch^4 is the value of the approximation error

Determine another integral value with further halving the step size (doubling the number of segments):

$$(I_{4n})_R = I_{4n} + \frac{I_{4n} - I_{2n}}{3}$$

Estimated true value is given by:

$$\begin{aligned} TV &\cong (I_{4n})_R + \frac{(I_{4n})_R - (I_{2n})_R}{15} \\ &= (I_{4n})_R + \frac{(I_{4n})_R - (I_{2n})_R}{4^{3-1} - 1} \end{aligned}$$

Romberg's method

A general expression for Romberg integration can be written as:

$$I_{k,j} = I_{k-1,j+1} + \frac{I_{k-1,j+1} - I_{k-1,j}}{4^{k-1} - 1}, k \geq 2$$

The index k represents the order of extrapolation

$k=1$ represents the values obtained from the regular trapezoidal rule

$k=2$ represents the values obtained using the true error estimate as $O(h^2)$

The value of an integral with a $j+1$ index is more accurate than the value of the integral with a j index

Romberg's method

Example 4:

The vertical distance covered by a rocket from $t_1=8$ s to $t_2=30$ s seconds is given by:

$$v(t) = 2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t$$

- a) Use Romberg's method to find the distance covered. Use the $n = 1, 2, 4,$ and 8-segment
- b) Find the absolute and relative true error

Table of results from a complex trapezoid rules for $n = 8$ segments

n	Δx	E_t	$\epsilon_t \%$	$\epsilon_a \%$
1	11868	-807	7.296	---
2	11266	-205	1.853	5.343
3	11153	-91.4	0.8265	1.019
4	11113	-51.5	0.4655	0.3594
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Romberg's method

From the table, the needed values from the original the trapezoidal rule are:

$$I_{1,1} = 11868 \qquad I_{1,2} = 11266$$

$$I_{1,3} = 11113 \qquad I_{1,4} = 11074$$

To get the $k=2$ order extrapolation values:

$$I_{k,j} = I_{k-1,j+1} + \frac{I_{k-1,j+1} - I_{k-1,j}}{4^{k-1} - 1}, k \geq 2$$

$$I_{2,1} = I_{1,2} + \frac{I_{1,2} - I_{1,1}}{3}$$

$$= 11266 + \frac{11266 - 11868}{3} = 11065$$

Romberg's method

Then

$$\begin{aligned} I_{2,2} &= I_{1,3} + \frac{I_{1,3} - I_{1,2}}{3} \\ &= 11113 + \frac{11113 - 11266}{3} \\ &= 11062 \end{aligned}$$

$$\begin{aligned} I_{2,3} &= I_{1,4} + \frac{I_{1,4} - I_{1,3}}{3} \\ &= 11074 + \frac{11074 - 11113}{3} \\ &= 11061 \end{aligned}$$

$$I_{k,j} = I_{k-1,j+1} + \frac{I_{k-1,j+1} - I_{k-1,j}}{4^{k-1} - 1}, k \geq 2$$

Romberg's method

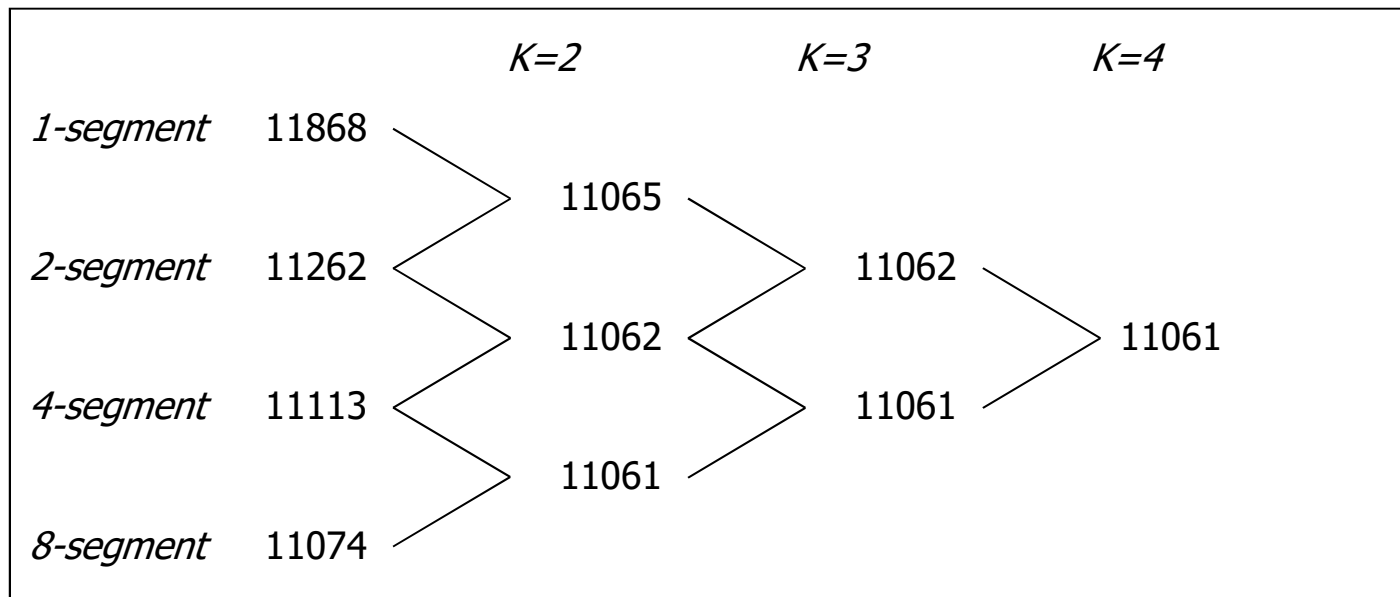
For the k=3 extrapolation values,

$$\begin{aligned} I_{3,1} &= I_{2,2} + \frac{I_{2,2} - I_{2,1}}{15} \\ &= 11062 + \frac{11062 - 11065}{15} \\ &= 11062 \end{aligned} \qquad \begin{aligned} I_{3,2} &= I_{2,3} + \frac{I_{2,3} - I_{2,2}}{15} \\ &= 11061 + \frac{11061 - 11062}{15} \\ &= 11061 \end{aligned}$$

For k=4

$$I_{4,1} = I_{3,2} + \frac{I_{3,2} - I_{3,1}}{63} = 11061 + \frac{11061 - 11062}{63} = 11061 m$$

Romberg's method



Improved estimates of the value of an integral using Romberg integration

Simpson's rule

The trapezoidal rule was based on approximating the integrand by a first order polynomial, and then integrating the polynomial over interval of integration. Simpson's rule is an extension of trapezoidal rule where the integrand is approximated by a second order polynomial.

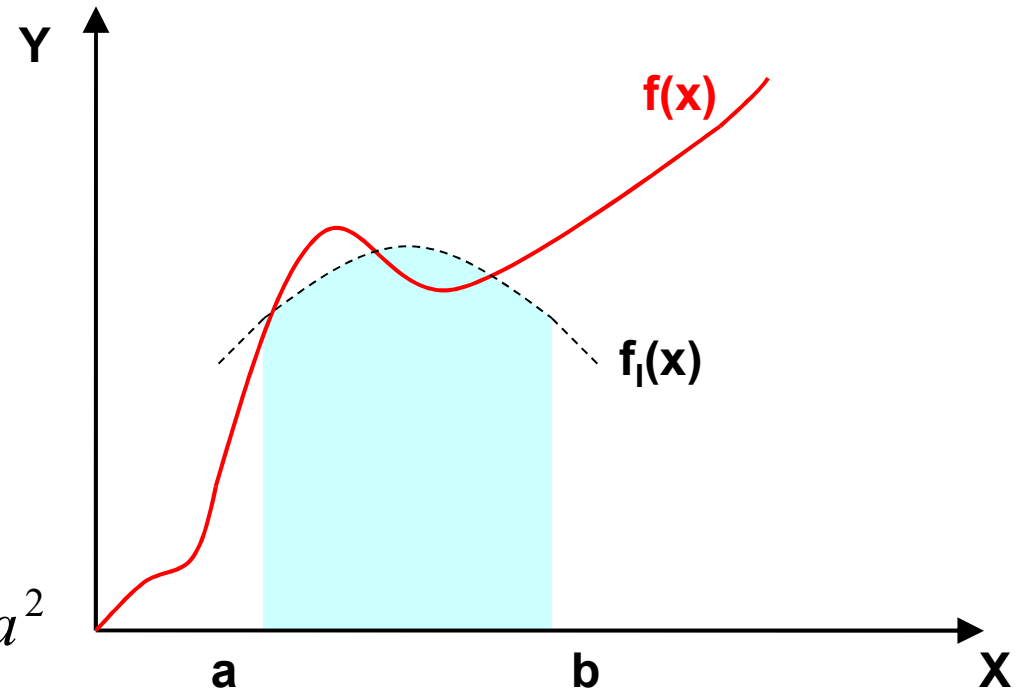
$$I = \int_a^b f(x) dx \approx \int_a^b f_2(x) dx$$

where:

$$f_2(x) = a_0 + a_1x + a_2x^2$$

Simpson's rule

$$\begin{aligned} &(a, f(a)), \\ &\left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right), \\ &(b, f(b)) \end{aligned}$$



$$\left\{ \begin{aligned} f(a) &= f_2(a) = a_0 + a_1 a + a_2 a^2 \\ f\left(\frac{a+b}{2}\right) &= f_2\left(\frac{a+b}{2}\right) = a_0 + a_1 \left(\frac{a+b}{2}\right) + a_2 \left(\frac{a+b}{2}\right)^2 \\ f(b) &= f_2(b) = a_0 + a_1 b + a_2 b^2 \end{aligned} \right.$$

Simpson's rule

$$a_0 = \frac{a^2 f(b) + abf(b) - 4abf\left(\frac{a+b}{2}\right) + abf(a) + b^2 f(a)}{a^2 - 2ab + b^2}$$

$$a_1 = - \frac{af(a) - 4af\left(\frac{a+b}{2}\right) + 3af(b) + 3bf(a) - 4bf\left(\frac{a+b}{2}\right) + bf(b)}{a^2 - 2ab + b^2}$$

$$a_2 = \frac{2\left(f(a) - 2f\left(\frac{a+b}{2}\right) + f(b)\right)}{a^2 - 2ab + b^2}$$

Simpson's rule

Since:

$$\begin{aligned} I &\approx \int_a^b f_2(x) dx \\ &= \int_a^b (a_0 + a_1 x + a_2 x^2) dx \\ &= \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} \right]_a^b \\ &= a_0(b-a) + a_1 \frac{b^2 - a^2}{2} + a_2 \frac{b^3 - a^3}{3} \end{aligned}$$

Simpson's rule

$$\int_a^b f_2(x) dx = \frac{b-a}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

$$h = \frac{b-a}{2}$$

It follows that:

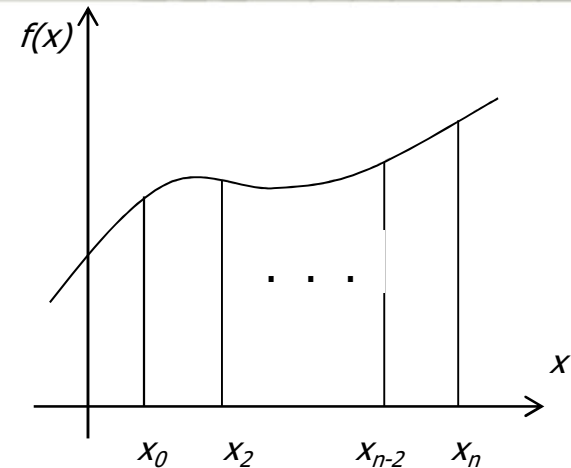
$$\int_a^b f_2(x) dx = \frac{h}{3} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

it is called Simpson's 1/3 rule

Complex Simpson's rule

$$\int_a^b f(x) dx = \int_{x_0}^{x_2} f(x) dx + \int_{x_2}^{x_4} f(x) dx + \dots$$

$$\dots + \int_{x_{n-4}}^{x_{n-2}} f(x) dx + \int_{x_{n-2}}^{x_n} f(x) dx$$



$$\int_a^b f(x) dx = (x_2 - x_0) \left[\frac{f(x_0) + 4f(x_1) + f(x_2)}{6} \right] + \dots$$

$$+ (x_4 - x_2) \left[\frac{f(x_2) + 4f(x_3) + f(x_4)}{6} \right] + \dots$$

$$\dots + (x_{n-2} - x_{n-4}) \left[\frac{f(x_{n-4}) + 4f(x_{n-3}) + f(x_{n-2})}{6} \right] + \dots$$

$$+ (x_n - x_{n-2}) \left[\frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6} \right]$$

$$x_i - x_{i-2} = 2h$$

$$i = 2, 4, \dots, n$$

Simpson's rule

$$\begin{aligned}\int_a^b f(x)dx &= 2h \left[\frac{f(x_0) + 4f(x_1) + f(x_2)}{6} \right] + \dots \\ &+ 2h \left[\frac{f(x_2) + 4f(x_3) + f(x_4)}{6} \right] + \dots \\ &+ 2h \left[\frac{f(x_{n-4}) + 4f(x_{n-3}) + f(x_{n-2})}{6} \right] + \dots \\ &+ 2h \left[\frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6} \right]\end{aligned}$$

Simpson's rule

$$\begin{aligned}\int_a^b f(x)dx &= \frac{h}{3} [f(x_0) + 4\{f(x_1) + f(x_3) + \dots + f(x_{n-1})\} + \dots] \\ &\quad \dots + 2\{f(x_2) + f(x_4) + \dots + f(x_{n-2})\} + f(x_n)] \\ &= \frac{h}{3} \left[f(x_0) + 4 \sum_{\substack{i=1 \\ i=\text{odd}}}^{n-1} f(x_i) + 2 \sum_{\substack{i=2 \\ i=\text{even}}}^{n-2} f(x_i) + f(x_n) \right] \\ &= \frac{b-a}{3n} \left[f(x_0) + 4 \sum_{\substack{i=1 \\ i=\text{odd}}}^{n-1} f(x_i) + 2 \sum_{\substack{i=2 \\ i=\text{even}}}^{n-2} f(x_i) + f(x_n) \right]\end{aligned}$$

Simpson's rule errors

Approximate values, for mentioned earlier example,
using Simpson's rule with multiple segments

n	Approximate values	E_t	$ \epsilon_t $
2	11065.72	4.38	0.0396%
4	11061.64	0.30	0.0027%
6	11061.40	0.06	0.0005%
8	11061.35	0.01	0.0001%
10	11061.34	0.00	0.0000%

Simpson's rule errors

Error for one segment $E_t = -\frac{(b-a)^5}{2880} f^{(4)}(\zeta), \quad a < \zeta < b$

Error for the multi-segment

$$E_1 = -\frac{(x_2 - x_0)^5}{2880} f^{(4)}(\zeta_1) = -\frac{h^5}{90} f^{(4)}(\zeta_1), \quad x_0 < \zeta_1 < x_2$$

$$E_2 = -\frac{(x_4 - x_2)^5}{2880} f^{(4)}(\zeta_2) = -\frac{h^5}{90} f^{(4)}(\zeta_2), \quad x_2 < \zeta_2 < x_4$$

$$E_i = -\frac{(x_{2i} - x_{2(i-1)})^5}{2880} f^{(4)}(\zeta_i) = -\frac{h^5}{90} f^{(4)}(\zeta_i), \quad x_{2(i-1)} < \zeta_i < x_{2i}$$

Simpson's rule errors

True error

$$E_t = \sum_{i=1}^{\frac{n}{2}} E_i = -\frac{h^5}{90} \sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i) = -\frac{(b-a)^5}{90n^5} \sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)$$

$$= -\frac{(b-a)^5}{90n^4} \frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$$

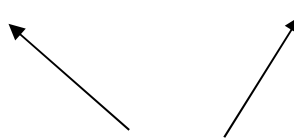
$$\overline{f}^{(4)} = \frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$$

$$E_t = -\frac{(b-a)^5}{90n^4} \overline{f}^{(4)}$$

the average value of the derivative

Gauss-Quadrature Method

Gaussian method is based on:

$$I = \int_a^b f(x) dx \approx c_1 f(x_1) + c_2 f(x_2)$$


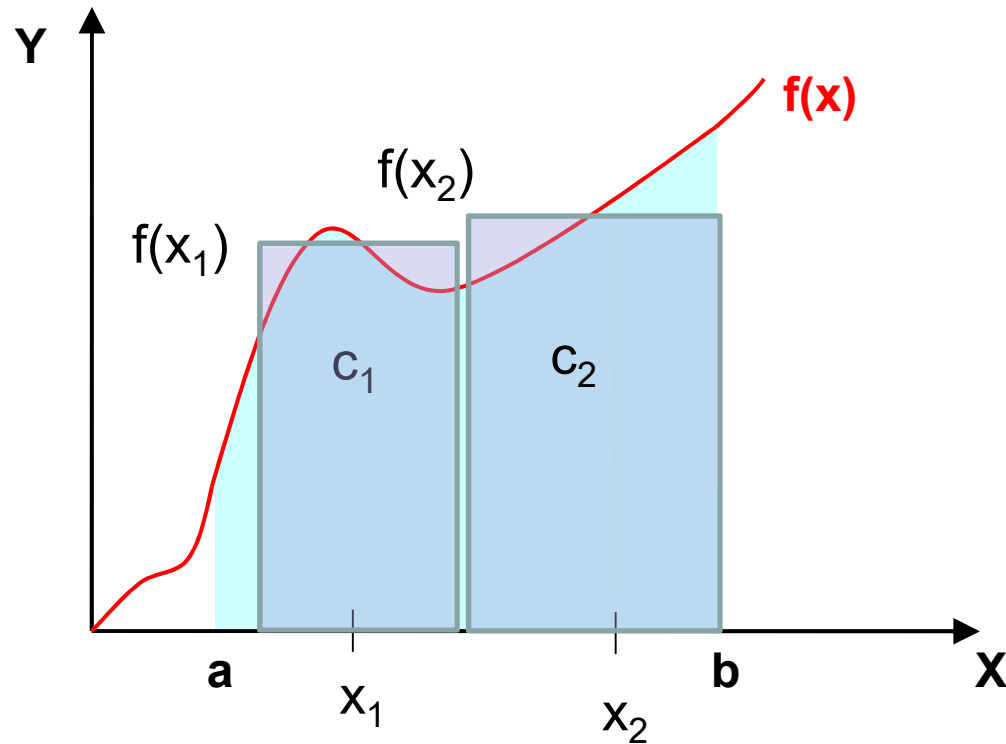
constant coefficients

Points x_1 and x_2 , which define the value of the integrand are not fixed (as before in the interval $\langle a, b \rangle$), but there are *a priori* distributed freely between $\langle a, b \rangle$.

Graphical representation of Gauss method

$$I = \int_a^b f(x) dx$$

$$\approx c_1 f(x_1) + c_2 f(x_2)$$



Gauss-Quadrature Method

There are four unknowns x_1, x_2, c_1, c_2 . These are found by assuming that the formula gives exact results for integrating a general third order polynomial:

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3.$$

$$\begin{aligned}\int_a^b f(x)dx &= \int_a^b (a_0 + a_1x + a_2x^2 + a_3x^3)dx \\ &= \left[a_0x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + a_3 \frac{x^4}{4} \right]_a^b \\ &= a_0(b-a) + a_1 \left(\frac{b^2 - a^2}{2} \right) + a_2 \left(\frac{b^3 - a^3}{3} \right) + a_3 \left(\frac{b^4 - a^4}{4} \right)\end{aligned}$$

Gauss-Quadrature Method

$$I = \int_a^b f(x) dx \approx c_1 f(x_1) + c_2 f(x_2)$$

Hence:

$$\begin{aligned} \int_a^b f(x) dx &= c_1 (a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3) + c_2 (a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_2^3) \\ &= a_0 (c_1 + c_2) + a_1 (c_1 x_1 + c_2 x_2) + a_2 (c_1 x_1^2 + c_2 x_2^2) + a_3 (c_1 x_1^3 + c_2 x_2^3) \end{aligned}$$

The formula would then give:

$$\begin{aligned} \int_a^b f(x) dx &= \int_a^b (a_0 + a_1 x + a_2 x^2 + a_3 x^3) dx \\ &= a_0 (b - a) + a_1 \left(\frac{b^2 - a^2}{2} \right) + a_2 \left(\frac{b^3 - a^3}{3} \right) + a_3 \left(\frac{b^4 - a^4}{4} \right) \end{aligned}$$

Gauss-Quadrature Method

$$a_0(c_1 + c_2) + a_1(c_1x_1 + c_2x_2) + a_2(c_1x_1^2 + c_2x_2^2) + a_3(c_1x_1^3 + c_2x_2^3)$$

$$= a_0(b - a) + a_1\left(\frac{b^2 - a^2}{2}\right) + a_2\left(\frac{b^3 - a^3}{3}\right) + a_3\left(\frac{b^4 - a^4}{4}\right)$$



$$b - a = c_1 + c_2$$

$$\frac{b^2 - a^2}{2} = c_1x_1 + c_2x_2$$

$$\frac{b^3 - a^3}{3} = c_1x_1^2 + c_2x_2^2$$

$$\frac{b^4 - a^4}{4} = c_1x_1^3 + c_2x_2^3$$

Gauss-Quadrature Method

This gives us four equations as follows:

$$\left\{ \begin{array}{l} b - a = c_1 + c_2 \\ \frac{b^2 - a^2}{2} = c_1 x_1 + c_2 x_2 \\ \frac{b^3 - a^3}{3} = c_1 x_1^2 + c_2 x_2^2 \\ \frac{b^4 - a^4}{4} = c_1 x_1^3 + c_2 x_2^3 \end{array} \right.$$

we can find that the above four simultaneous nonlinear equations have only one acceptable solution

$$\begin{aligned} x_1 &= \left(\frac{b-a}{2} \right) \left(-\frac{1}{\sqrt{3}} \right) + \frac{b+a}{2} & x_2 &= \left(\frac{b-a}{2} \right) \left(\frac{1}{\sqrt{3}} \right) + \frac{b+a}{2} \\ c_1 &= \frac{b-a}{2} & c_2 &= \frac{b-a}{2} \end{aligned}$$

Gauss-Quadrature Method

Hence:

$$\begin{aligned}\int_a^b f(x)dx &\approx c_1 f(x_1) + c_2 f(x_2) \\ &= \frac{b-a}{2} f\left(\frac{b-a}{2}\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right) + \frac{b-a}{2} f\left(\frac{b-a}{2}\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right)\end{aligned}$$

General n-point rules would approximate the integral:

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) + \dots + c_n f(x_n)$$

Gauss-Quadrature Method

The coefficients and arguments given for n-point Gauss quadrature rule are given for integrals of the form in the $[-1,1]$ interval:

$$\int_{-1}^1 g(x) dx \cong \sum_{i=1}^n c_i g(x_i)$$

n	coefficients	Function arguments
2	$c_1 = 1.000000000$ $c_2 = 1.000000000$	$x_1 = -0.577350269$ $x_2 = 0.577350269$
3	$c_1 = 0.555555556$ $c_2 = 0.888888889$ $c_3 = 0.555555556$	$x_1 = -0.774596669$ $x_2 = 0.000000000$ $x_3 = 0.774596669$
4	$c_1 = 0.347854845$ $c_2 = 0.652145155$ $c_3 = 0.652145155$ $c_4 = 0.347854845$	$x_1 = -0.861136312$ $x_2 = -0.339981044$ $x_3 = 0.339981044$ $x_4 = 0.861136312$

Gauss-Quadrature Method

n	coefficients	Function arguments
5	$c_1 = 0.236926885$ $c_2 = 0.478628670$ $c_3 = 0.568888889$ $c_4 = 0.478628670$ $c_5 = 0.236926885$	$x_1 = -0.906179846$ $x_2 = -0.538469310$ $x_3 = 0.000000000$ $x_4 = 0.538469310$ $x_5 = 0.906179846$
6	$c_1 = 0.171324492$ $c_2 = 0.360761573$ $c_3 = 0.467913935$ $c_4 = 0.467913935$ $c_5 = 0.360761573$ $c_6 = 0.171324492$	$x_1 = -0.932469514$ $x_2 = -0.661209386$ $x_3 = -0.2386191860$ $x_4 = 0.2386191860$ $x_5 = 0.661209386$ $x_6 = 0.932469514$

Gauss-Quadrature Method

So if the table is given for: $\int_{-1}^1 g(x) dx$ integrals, how does one solve

$$\int_a^b f(x) dx \quad ?$$

The answer lies in that any integral with limits of $[a, b]$

can be converted into an integral with limits: $[-1, 1]$

$$\text{Let } x = mt + c$$

$$\text{if } x = a, \quad t = -1$$

$$\text{ir } x = b, \quad t = 1$$

It follows that:

$$\left\{ \begin{array}{l} m = \frac{b-a}{2} \\ c = \frac{b+a}{2} \end{array} \right.$$

Gauss-Quadrature Method

Hence:
$$x = \frac{b-a}{2}t + \frac{b+a}{2}$$

$$dx = \frac{b-a}{2}dt$$

Substituting our values of x and dx into the integral gives us:

$$\int_a^b f(x)dx = \int_{-1}^1 f\left(\frac{b-a}{2}t + \frac{b+a}{2}\right) \frac{b-a}{2} dt$$

Gauss-Quadrature Method

Example 5:

Use two-point Gauss quadrature rule to approximate the distance covered by a rocket from $t = 8$ s to $t = 30$ s as given by:

$$v(t) = 2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t$$

- Use the Gauss-Quadrature Method to find the distance covered from $t_1 = 8$ s to $t_2 = 30$ s
- Find the true error.

Gauss-Quadrature Method

First, change the limits of integration from from [8,30] to [-1,1]

$$\begin{aligned}\int_8^{30} f(t) dt &= \frac{30-8}{2} \int_{-1}^1 f\left(\frac{30-8}{2}x + \frac{30+8}{2}\right) dx \\ &= 11 \int_{-1}^1 f(11x + 19) dx\end{aligned}$$

The weighting factors and function argument values are (n=2):

$$c_1 = 1.000000000$$

$$x_1 = -0.577350269$$

$$c_2 = 1.000000000$$

$$x_2 = 0.577350269$$

Gauss-Quadrature Method

The formula is:

$$\begin{aligned} 11 \int_{-1}^1 f(11x + 19) dx &\approx 11c_1 f(11x_1 + 19) + 11c_2 f(11x_2 + 19) \\ &= 11f(11(-0.5773503) + 19) + 11f(11(0.5773503) + 19) \\ &= 11f(12.64915) + 11f(25.35085) \\ &= 11(296.8317) + 11(708.4811) \\ &= 11058.44 \text{ m} \end{aligned}$$

Gauss-Quadrature Method

Since:

$$f(12.64915) = 2000 \ln \left[\frac{140000}{140000 - 2100(12.64915)} \right] - 9.8(12.64915)$$

$$= 296.8317$$

$$f(25.35085) = 2000 \ln \left[\frac{140000}{140000 - 2100(25.35085)} \right] - 9.8(25.35085)$$

$$= 708.4811$$

Gauss-Quadrature Method

b) The absolute relative true error: E_t

$$\begin{aligned} E_t &= 11061.34 - 11058.44 \\ &= 2.9000 \text{ m} \end{aligned}$$

c) The relative error: $|\epsilon_t|$

$$\begin{aligned} |\epsilon_t| &= \left| \frac{11061.34 - 11058.44}{11061.34} \right| \times 100\% \\ &= 0.0262\% \end{aligned}$$