NUMERICAL METHODS

Lecture 5.

Set of linear equations

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OUTLINE

- Exact methods
- Naïve Gaussian Elimination
- Gauss-Siedel Method
- LU Decomposition

System of linear equations

Consider the system of linear equations *m* with *n* unknowns in the form of

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}\right\}$$

with coefficients a_{ik} and b_i that belong to K (*K* = *R* or *K* = *C*)

System of linear equations

The system of linear equations:

$$\mathbf{Ax} = \mathbf{b}$$

where:

- **A** – matrix with m rows and n columns
- **x** – vector with n unknowns
- **b** – vector of m known numbers

possible solutions:

- Infinitely many solutions
- Exactly one solution
- No solution (inconsistent system)

Kronecker-Capelli theorem

Matrix system of equations is called a coefficient matrix *A*

$$A = \begin{bmatrix} a_{11} \dots a_{1n} \\ \dots\dots\dots \\ a_{m1} \dots a_{mn} \end{bmatrix} \qquad C = \begin{bmatrix} a_{11} \dots a_{1n} b_1 \\ \dots\dots\dots \\ a_{m1} \dots a_{mn} b_m \end{bmatrix}$$

Expanded matrix is called the matrix *C*, also referred to as *A* / *B*, formed from the matrix *A* by adding to it a column of free terms

Kronecker-Capelli theorem

System of m linear equations with n unknowns has a solution if the rank r of the main matrix equals to the rank of the expanded matrix :

$$\textit{rank } A = \textit{rank } C = r$$

For any matrix, the rank is equal to r if and only if there exists a nonzero minor of rank k of this matrix, and every minor of rank larger than k is zero.

Kronecker-Capelli theorem

If the rank r of both matrices is equal to the number of unknowns, there is one solution of the system of equations , i.e., one set of numbers satisfying the equation can be found; a ***system is consistent***

$$rank\ A = rank\ C = n$$

Kronecker-Capelli theorem

If a common order r the both matrices is smaller than the number of unknowns n,

$$\text{rank } A = \text{rank } C < n$$

then the system has no unique solution, it depends on (n - r) parameters

Thus, (n - r) unknowns can be chosen arbitrarily, and the remaining r unknowns can be uniquely determined from the matrix equation.

Kronecker-Capelli theorem

If the rank r of the main matrix is smaller than the rank of an extended matrix,

rank A < rank C

the system of linear equations has no solution; the **system *is inconsistent***

Exact methods - definition

If the solution of equations **Ax=b** is obtained by the transformation of **A** and **B**, assuming exactly performed arithmetic operations, after a finite number of actions we get a solution.

Exact methods - features

- A small number of calculations needed to determine the solution
- If the task is ill-conditioned numerically, the solution can be subject to a significant error.
- They can be unstable due to rounding off errors
- The transformation matrix **A** uses to a large extent the machine's memory, especially if the original data **A** and **b** should be kept to a final verification

Exact methods - example

<u>*Example – Cramer's formulas*</u>

Method 1:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$

$$x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{21}a_{12}} \qquad x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{21}a_{12}}$$

We assume accurate to two decimal digits each result before further calculation is rounded

$$\begin{cases} 0.99x_1 + 0.70x_2 = 0.54 \\ 0.70x_1 + 0.50x_2 = 0.38 \end{cases}$$

$$a_{11}a_{22} = 0.99 \cdot 0.50 = 0.4950 = 0.50$$

$$a_{21}a_{12} = 0.70 \cdot 0.70 = 0.4900 = 0.49$$

$$a_{11}a_{22} - a_{21}a_{12} = 0.50 - 0.49 = 0.01$$

Exact methods - example

$$a_{22}b_1 - a_{12}b_2 = 0.50 \cdot 0.54 - 0.70 \cdot 0.38$$

$$= 0.2700 - 0.2660 = 0.27 - 0.27 = 0$$

$$a_{11}b_2 - a_{21}b_1 = 0.99 \cdot 0.38 - 0.70 \cdot 0.54$$

$$= 0.3762 - 0.3780 = 0.38 - 0.38 = 0$$

$$x_1 = \frac{0}{0.01} = 0$$

$$x_2 = \frac{0}{0.01} = 0$$

The exact solution to this system of equations gives the result:

$$x_1 = 0.80 \qquad x_2 = -0.36$$

Exact methods - example

Method 2: Gaussian elimination method

$$0.99x_1 + 0.70x_2 = 0.54$$
$$0.70x_1 + 0.50x_2 = 0.38$$

We eliminate the unknown x_1 from the second equation of the system of equations. For this purpose multiplying the first equation by:

$$\frac{a_{21}}{a_{11}} = \frac{0.70}{0.99} = 0.7070 \cong 0.71$$

We receive:

$$0.70x_1 + 0.4949x_2 = 0.3818$$
$$0.70x_1 + 0.50x_2 = 0.38$$

By subtracting the equations, after having rounded to two digits:

$$0.00x_2 = 0.00$$

The system has infinitely many solutions.

Sets of equations with triangular matrix

Triangular matrix – definition

Triangular matrix is called lower (upper) triangular matrix, where all the elements of above (under) diagonal are equal to zero.

$$\mathbf{L} = \begin{bmatrix} l_{1,1} & 0 & 0 & 0 & 0 \\ l_{2,1} & l_{2,2} & 0 & 0 & 0 \\ l_{3,1} & l_{3,2} & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ l_{n,1} & l_{n,2} & \ldots & l_{n,n-1} & l_{n,n} \end{bmatrix} \qquad \mathbf{U} = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & \ldots & u_{1,n} \\ 0 & u_{2,2} & u_{2,3} & \ldots & u_{2,n} \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & u_{n-1,n} \\ 0 & 0 & 0 & 0 & u_{n,n} \end{bmatrix}$$

lower triangular matrix *upper triangular matrix*

Sets of equations with triangular matrix

Calculating the determinant of a triangular matrix comes down to the multiplication of elements lying on the main diagonal:

$$\mathbf{L} = \begin{bmatrix} l_{1,1} & 0 & 0 & 0 & 0 \\ l_{2,1} & l_{2,2} & 0 & 0 & 0 \\ l_{3,1} & l_{3,2} & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ l_{n,1} & l_{n,2} & \dots & l_{n,n-1} & l_{n,n} \end{bmatrix}$$

$$\det(L) = \prod_{i=1}^{n} l_{i,i} = l_{1,1} \cdot l_{2,2} \cdot \ldots l_{n,n}$$

$$\mathbf{U} = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & \dots & u_{1,n} \\ 0 & u_{2,2} & u_{2,3} & \dots & u_{2,n} \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & u_{n-1,n} \\ 0 & 0 & 0 & 0 & u_{n,n} \end{bmatrix}$$

$$\det(U) = \prod_{i=1}^{n} u_{i,i} = u_{1,1} \cdot u_{2,2} \cdot \ldots u_{n,n}$$

Sets of equations with triangular matrix

If the matrix **A** system of n equations with n unknowns **Ax=b** is a triangular matrix (upper or lower), then the solution **x** of this system of equations can be obtained by performing a small number of arithmetic operations with small rounding errors

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1,n-1}x_{n-1} + a_{1n}x_n = b_1$$

$$a_{22}x_2 + \ldots + a_{2,n-1}x_{n-1} + a_{2n}x_n = b_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n = b_{n-1}$$

$$a_{nn}x_n = b_n \;\Rightarrow\; x_n = \frac{b_n}{a_{nn}}$$

Generally

$$x_i = \frac{b_i - a_{in}x_n - \ldots - a_{ii+1}x_{i+1}}{a_{ii}} \qquad i = n-1,\ n-2,\ \ldots,\ 1$$

Calculation cost:

To determine the vector **x,** there should be performed M multiplications and divisions and D additions:

$$M = \frac{1}{2}n^2 + \frac{1}{2}n$$

$$D = \frac{1}{2}n^2 + \frac{1}{2}n$$

Gaussian Elimination

The first phase (the phase of the elimination of the "forward„ coefficients)

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n$$

n-1 steps of elimination are required

Gaussian Elimination

Step 1. From the second row subtract the first one, divided by a_{11} and multiplied by a_{21}

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1 \quad \Bigg| \; \frac{a_{21}}{a_{11}}$$

$$\Downarrow$$

$$\begin{cases} a_{21}x_1 + \dfrac{a_{21}}{a_{11}}a_{12}x_2 + \ldots + \dfrac{a_{21}}{a_{11}}a_{1n}x_n = \dfrac{a_{21}}{a_{11}}b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2 \end{cases}$$

Result:

$$\left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)x_2 + \ldots + \left(a_{2n} - \frac{a_{21}}{a_{11}}a_{1n}\right)x_n = b_2 - \frac{a_{21}}{a_{11}}b_1$$

Gaussian Elimination

Similarly, acting with subsequent rows:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1 \\ \qquad a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2 \\ \qquad a'_{32}x_2 + a'_{33}x_3 + \ldots + a'_{3n}x_n = b'_3 \\ \qquad \cdots\cdots\cdots\cdots\cdots\cdots \\ \qquad a'_{n2}x_2 + a'_{n3}x_3 + \ldots + a'_{nn}x_n = b'_n \end{cases}$$

where: $a'_{22} = a_{22} - \frac{a_{21}}{a_{11}} a_{12}$

$$\vdots$$

$$a'_{2n} = a_{2n} - \frac{a_{21}}{a_{11}} a_{1n}$$

Gaussian Elimination

Step 2. We repeat the procedure step 1 for the third row

$$a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2 \qquad \left| \frac{a'_{32}}{a'_{22}} \right.$$

$$\Downarrow$$

$$\begin{cases} a'_{32}x_2 + a'_{23}\dfrac{a'_{32}}{a'_{22}}x_3 + \ldots + a'_{2n}\dfrac{a'_{32}}{a'_{22}}x_n = \dfrac{a'_{32}}{a'_{22}}b'_2 \\ a'_{32}x_2 + a'_{33}x_3 + \ldots + a'_{3n}x_n = b'_3 \end{cases}$$

Result:

$$\left(a'_{33} - \frac{a'_{32}}{a'_{22}}a'_{23} \right)x_3 + \ldots + \left(a'_{3n} - \frac{a'_{32}}{a'_{22}}a'_{2n} \right)x_n = b'_3 - \frac{a'_{32}}{a'_{22}}b'_2$$

Gaussian Elimination

After second step we get:

$$a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2$$

$$a''_{33}x_3 + \ldots + a''_{3n}x_n = b''_3$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a''_{n3}x_3 + \ldots + a''_{nn}x_n = b''_n$$

Gaussian Elimination

At the end of step n-1 the set of equations takes the form:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2$$

$$a''_{33}x_3 + \ldots + a''_{3n}x_n = b''_3$$

....

$$a_{nn}^{(n-1)}x_n = b_n^{(n-1)}$$

Gaussian Elimination

After the n-1 elimination step of variables the resulting equations can be written in a matrix form:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & 0 & a''_{33} & \cdots & a''_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & a_{nn}^{(n-1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ \vdots \\ b_n^{(n-1)} \end{bmatrix}$$

The resulting matrix is a triangular matrix!

The second phase – so called back-substitution

Start with the last equation because it has only one unknown

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

$$x_i = \frac{b_i^{(i-1)} - a_{i,i+1}^{(i-1)} x_{i+1} - a_{i,i+2}^{(i-1)} x_{i+2} - \ldots - a_{i,n}^{(i-1)} x_n}{a_{ii}^{(i-1)}} \quad for\ i = n-1,\ldots,1$$

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}} \quad for\ \ i = n-1,\ldots,1$$

Gaussian Elimination– computational cost

The total number of multiplications and divisions:

$$M = \frac{1}{3}n^3 + n^2 - \frac{1}{3}n$$

The total number of additions:

$$D = \frac{1}{3}n^3 + \frac{1}{2}n^2 - \frac{5}{6}n$$

Gaussian Elimination - example

Example:

Time t (s)	Velocity (m/s)
5	106.8
8	177.2
12	279.2

The velocity data is approximated by a polynomial as:

$$v(t) = a_1 t^2 + a_2 t + a_3 , \qquad 5 \le \mathrm{t} \le 12.$$

Find the velocity at t=6 seconds.

Gaussian Elimination - example

$$v(t) = a_1 t^2 + a_2 t + a_3, \quad 5 \le t \le 12.$$

$$\begin{bmatrix} t_1^2 & t_1 & 1 \\ t_2^2 & t_2 & 1 \\ t_3^2 & t_3 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$t_1 = 5\,s, v(5) = 106{,}8\ m/s$$

$$t_2 = 8\,s, v(5) = 177{,}2\ m/s$$

$$t_3 = 12\,s, v(5) = 279{,}2\ m/s$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Gaussian Elimination - example

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

Divide equation 1 by 25 and multiply by 64 $\frac{64}{25} = 2.56$

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix} \times 2.56 = \begin{bmatrix} 64 & 12.8 & 2.56 & \vdots & 273.408 \end{bmatrix}$$

Subtract the result of equation 2

$$\begin{array}{r} \begin{bmatrix} 64 & 8 & 1 & \vdots & 177.2 \end{bmatrix} \\ -\begin{bmatrix} 64 & 12.8 & 2.56 & \vdots & 273.408 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & -4.8 & -1.56 & \vdots & -96.208 \end{bmatrix} \end{array}$$

We receive :

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

Gaussian Elimination - example

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

Divide equation 1 by 25 and multiply by 144 $\frac{144}{25} = 5.76$

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix} \times 5.76 = \begin{bmatrix} 144 & 28.8 & 5.76 & \vdots & 615.168 \end{bmatrix}$$

Subtract the result of equation 3

$$\begin{array}{r} \begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix} \\ -\begin{bmatrix} 144 & 28.8 & 5.76 & \vdots & 615.168 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & -16.8 & -4.76 & \vdots & -335.968 \end{bmatrix} \end{array}$$

After the first step of elimination

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 0 & -16.8 & -4.76 & \vdots & -335.968 \end{bmatrix}$$

Gaussian Elimination - example

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 0 & -16.8 & -4.76 & \vdots & -335.968 \end{bmatrix}$$

Divide equation 2 by -4.8 and multiply by -16.8

$$\frac{-16.8}{-4.8} = 3.5$$

$$\begin{bmatrix} 0 & -4.8 & -1.56 & \vdots & -96.208 \end{bmatrix} \times 3.5 = \begin{bmatrix} 0 & -16.8 & -5.46 & \vdots & -336.728 \end{bmatrix}$$

Subtract the result of equation 3

$$\begin{array}{r} \begin{bmatrix} 0 & -16.8 & -4.76 & \vdots & 335.968 \end{bmatrix} \\ -\begin{bmatrix} 0 & -16.8 & -5.46 & \vdots & -336.728 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 0 & 0.7 & \vdots & 0.76 \end{bmatrix} \end{array}$$

After the second step of elimination

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 0 & 0 & 0.7 & \vdots & 0.76 \end{bmatrix}$$

Gaussian Elimination - example

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.2 \\ 0 & 0 & 0.7 & \vdots & 0.7 \end{bmatrix} \Rightarrow \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.208 \\ 0.76 \end{bmatrix}$$

Back Substitution

Solving for a_3

$$0.7a_3 = 0.76$$

$$a_3 = \frac{0.76}{0.7}$$

$$a_3 = 1.08571$$

Gaussian Elimination - example

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.208 \\ 0.76 \end{bmatrix}$$

Solving for a_2

$$-4.8a_2 - 1.56a_3 = -96.208$$

$$a_2 = \frac{-96.208 + 1.56a_3}{-4.8} \qquad a_3 = 1.08571$$

$$a_2 = \frac{-96.208 + 1.56 \times 1.08571}{-4.8}$$

$$a_2 = 19.6905$$

Gaussian Elimination - example

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.2 \\ 0.76 \end{bmatrix}$$

$$a_3 = 1.08571 \qquad a_2 = 19.6905$$

Solving for a_1

$$25a_1 + 5a_2 + a_3 = 106.8$$

$$a_1 = \frac{106.8 - 5a_2 - a_3}{25}$$

$$= \frac{106.8 - 5 \times 19.6905 - 1.08571}{25}$$

$$= 0.290472$$

Gaussian Elimination - example

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.290472 \\ 19.6905 \\ 1.08571 \end{bmatrix}$$

$$v(t) = a_1 t^2 + a_2 t + a_3 = 0.290472 t^2 + 19.6905 t + 1.08571, \quad 5 \le t \le 12$$

$$v(6) = 0.290472(6)^2 + 19.6905(6) + 1.08571 = 129.686 \text{ m/s.}$$

Gaussian Elimination

Disadvantages of the method:

- May stop the process of calculation due to a division by zero.
- It is particularly susceptible to accumulation of rounding errors.

Advantages of the method:

- The number of operations in the method of Gaussian elimination is much smaller than in the method of Cramer

In case of 15 equations

M=1345 multiplications in the method of Gaussian Elimination and $M=5\cdot10^{12}$ for Cramer formulas

Digital machine performs 10^6 multiplications per second: 0,01 s in method of Gaussian Elimination and more than a year to formulas Cramer

Gaussian Elimination

Division by zero can occur any time during elimination of variables

$$\begin{bmatrix} 12 & 10 & -7 \\ 6 & 5 & 3 \\ 24 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 14 \\ 28 \end{bmatrix} \longrightarrow \begin{bmatrix} 12 & 10 & -7 \\ 0 & 0 & 6.5 \\ 0 & -21 & 19 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 6.5 \\ -2 \end{bmatrix}$$

In the next step, division by zero

Gaussian Elimination

The system of equations:

$$\begin{bmatrix} 20 & 15 & 10 \\ -3 & -2.249 & 7 \\ 5 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 1.751 \\ 9 \end{bmatrix}$$

Exact solution	The solution with an accuracy 6 decimal digits at each step	The solution with an accuracy 5 decimal digits at each step
$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.9625 \\ 1.05 \\ 0.999995 \end{bmatrix}$	$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.625 \\ 1.5 \\ 0.99995 \end{bmatrix}$

Gaussian Elimination

Partial pivoting method

- with a partial choice of the base element

- Prevents division by zero
- Reduces the numerical error

The basic element is called the element of matrix A, with which eliminates the variable of further equations. So far as the basic elements chose the element lying on the diagonal

$$a_{kk}$$

Using partial choice the base element to select of the elements of the k-th column in the k-th matrix, which has the largest module. By changing sequence of rows in the matrix, you can get the basic element lying on the diagonal

Gaussian Elimination

<u>*Example :*</u>

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

The values in the first column are: $|25|, |64|, |144|$

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix} \Rightarrow \begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix}$$

Exchange the third row and the first row

Gaussian Elimination

Calculate the determinant of a matrix $[A]$

$$[A]=\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

After Gauss elimination

$$[B]=\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Useful Theorem: If the matrix B is formed from the matrix A by adding or subtracting from one row another row multiplied by the number, the determinant remains the same

det(A)=det(B)=25 (-4,8) (0.7)=-84,00

Gaussian Elimination

Following application of the method with partial selection of basic element , we have received matrix[C]

$$[C]=\begin{bmatrix} 144 & 12 & 1 \\ 0 & 2.917 & 0.8264 \\ 0 & 0 & -0.2 \end{bmatrix}$$

Useful Theorem: If the matrix B is formed from the matrix A by exchanging one row with another, then a change of sign of the determinant occurs

det(C)=(-)(-)det(B)=144 (2.917) (-0.2)=-84,00

Gaussian Elimination

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix}$$

Divide row 1 by 144 and multiply by 64

$$\frac{64}{144} = 0.4444$$

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix} \times 0.4444 = \begin{bmatrix} 63.99 & 5.333 & 0.4444 & \vdots & 124.1 \end{bmatrix}$$

Subtract the result of equation 2

$$\begin{array}{r} \begin{bmatrix} 64 & 8 & 1 & \vdots & 177.2 \end{bmatrix} \\ -\begin{bmatrix} 63.99 & 5.333 & 0.4444 & \vdots & 124.1 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 2.667 & 0.5556 & \vdots & 53.10 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \\ 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix}$$

Gaussian Elimination

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \\ 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix}$$

Divide equation 1 by 144 and multiply by 25 $\quad \dfrac{25}{144} = 0.1736$

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix} \times 0.1736 = \begin{bmatrix} 25.00 & 2.083 & 0.1736 & \vdots & 48.47 \end{bmatrix}$$

Subtract the result of equation 3

$$\begin{array}{r} \begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix} \\ -\begin{bmatrix} 25 & 2.083 & 0.1736 & \vdots & 48.47 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 2.917 & 0.8264 & \vdots & 58.33 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \end{bmatrix}$$

Gaussian Elimination

The values in the second column of the second and third row is:

$$|2.667|,|2.917|$$

Maximum is 2.917 in the third row

Replace the third row of the second

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \end{bmatrix} \Rightarrow \begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \end{bmatrix}$$

Gaussian Elimination

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \end{bmatrix}$$

Divide equation 2 by 2.917 and multiply by 2.667

$$\frac{2.667}{2.917} = 0.9143.$$

$$\begin{bmatrix} 0 & 2.917 & 0.8264 & \vdots & 58.33 \end{bmatrix} \times 0.9143 = \begin{bmatrix} 0 & 2.667 & 0.7556 & \vdots & 53.33 \end{bmatrix}$$

Subtract the result of equation 3

$$\begin{array}{r} \begin{bmatrix} 0 & 2.667 & 0.5556 & \vdots & 53.10 \end{bmatrix} \\ -\begin{bmatrix} 0 & 2.667 & 0.7556 & \vdots & 53.33 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 0 & -0.2 & \vdots & -0.23 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \\ 0 & 0 & -0.2 & \vdots & -0.23 \end{bmatrix}$$

Gaussian Elimination

$$\begin{bmatrix} 144 & 12 & 1 \\ 0 & 2.917 & 0.8264 \\ 0 & 0 & -0.2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 279.2 \\ 58.33 \\ -0.23 \end{bmatrix}$$

Solving for a_2

$$2.917a_2 + 0.8264a_3 = 58.33$$

$$a_2 = \frac{58.33 - 0.8264a_3}{2.917}$$

$$= \frac{58.33 - 0.8264 \times 1.15}{2.917}$$

$$= 19.67$$

Gaussian Elimination

$$\begin{bmatrix} 144 & 12 & 1 \\ 0 & 2.917 & 0.8264 \\ 0 & 0 & -0.2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 279.2 \\ 58.33 \\ -0.23 \end{bmatrix}$$

Solving for a_1

$$144a_1 + 12a_2 + a_3 = 279.2$$

$$a_1 = \frac{279.2 - 12a_2 - a_3}{144}$$

$$= \frac{279.2 - 12 \times 19.67 - 1.15}{144}$$

$$= 0.2917$$

Gaussian Elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.2917 \\ 19.67 \\ 1.15 \end{bmatrix}$$

Gauss-Seidel Method

System of n equations with n unknowns:

$$
\begin{cases}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2 \\
\vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n
\end{cases}
$$

Gauss-Seidel Method

The transformation equations to the form:

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3 \ldots\ldots - a_{1n}x_n}{a_{11}}$$ ← from equation 1

$$x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3 \ldots\ldots - a_{2n}x_n}{a_{22}}$$ ← from equation 2

$$\vdots \qquad \vdots \qquad \vdots$$

with n-1

$$x_{n-1} = \frac{b_{n-1} - a_{n-1,1}x_1 - a_{n-1,2}x_2 \ldots\ldots - a_{n-1,n-2}x_{n-2} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$

$$x_n = \frac{b_n - a_{n1}x_1 - a_{n2}x_2 - \ldots\ldots - a_{n,n-1}x_{n-1}}{a_{nn}}$$ ← from equation n

Gauss-Seidel Method

General form of the i - th equation

$$x_i = \frac{b_i - \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j}{a_{ii}}, i = 1, 2, \ldots, n.$$

This is an iterative method

Gauss-Seidel Method

We assume the initial values of x_1 to x_n and substitute them into previously transformed equations

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

Calculate the absolute value of the relative approximate error

$$\left|\in_a\right|_i = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right| \times 100$$

The iterations are stopped when the absolute value of the relative approximate error is less than a prespecified tolerance for all unknowns.

Gauss-Seidel Method

Example:

Time t (s)	Velocity (m/s)
5	106.8
8	177.2
12	279.2

The velocity data is approximated by a polynomial as:

$$v(t) = a_1 t^2 + a_2 t + a_3 \, , \qquad 5 \leq t \leq 12.$$

Find coefficients a_1, a_2, a_3 by Gauss-Seidel method

and velocity in time t = 6 s

Gauss-Seidel Method

Using a Matrix template of the form:

$$\begin{bmatrix} t_1^2 & t_1 & 1 \\ t_2^2 & t_2 & 1 \\ t_3^2 & t_3 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

The system of equations becomes:

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Initial Guess: Assume an initial guess of

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

Gauss-Seidel Method

Rewriting each equation:

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$a_1 = \frac{106.8 - 5a_2 - a_3}{25}$$

$$a_2 = \frac{177.2 - 64a_1 - a_3}{8}$$

$$a_3 = \frac{279.2 - 144a_1 - 12a_2}{1}$$

The first iteration:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$$a_1 = \frac{106.8 - 5(2) - (5)}{25} = 3.6720$$

$$a_2 = \frac{177.2 - 64(3.6720) - (5)}{8} = -7.8510$$

$$a_3 = \frac{279.2 - 144(3.6720) - 12(-7.8510)}{1} = -155.36$$

Gauss-Seidel Method

Finding the absolute relative approximate error:

$$\left|\in_a\right|_i = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right| \times 100$$

$$\left|\in_a\right|_1 = \left|\frac{3.6720 - 1.0000}{3.6720}\right| \times 100 = 72.76\%$$

$$\left|\in_a\right|_2 = \left|\frac{-7.8510 - 2.0000}{-7.8510}\right| \times 100 = 125.47\%$$

$$\left|\in_a\right|_3 = \left|\frac{-155.36 - 5.0000}{-155.36}\right| \times 100 = 103.22\%$$

At the end of the first iteration:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 3.6720 \\ -7.8510 \\ -155.36 \end{bmatrix}$$

The maximum relative approximate error is 125.47%

Gauss-Seidel Method

Second iteration:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 3.6720 \\ -7.8510 \\ -155.36 \end{bmatrix}$$

The results of the first iteration:

$$a_1 = \frac{106.8 - 5(-7.8510) - 155.36}{25} = 12.056$$

$$a_2 = \frac{177.2 - 64(12.056) - 155.36}{8} = -54.882$$

$$a_3 = \frac{279.2 - 144(12.056) - 12(-54.882)}{1} = -798.34$$

Finding the absolute relative approximate error:

$$\left|\in_a\right|_1 = \left|\frac{12.056 - 3.6720}{12.056}\right| x100 = 69.543\%$$

$$\left|\in_a\right|_2 = \left|\frac{-54.882 - (-7.8510)}{-54.882}\right| x100 = 85.695\%$$

$$\left|\in_a\right|_3 = \left|\frac{-798.34 - (-155.36)}{-798.34}\right| x100 = 80.540\%$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 12.056 \\ -54.882 \\ -798.54 \end{bmatrix}$$

The maximum relative approximate error is 85.695%

Gauss-Seidel Method

Iteration	a_1	$\left|\in_a\right|_1 \%$	a_2	$\left|\in_a\right|_2 \%$	a_3	$\left|\in_a\right|_3 \%$
1	3.6720	72.767	−7.8510	125.47	−155.36	103.22
2	12.056	69.543	−54.882	85.695	−798.34	80.540
3	47.182	74.447	−255.51	78.521	−3448.9	76.852
4	193.33	75.595	−1093.4	76.632	−14440	76.116
5	800.53	75.850	−4577.2	76.112	−60072	75.963
6	3322.6	75.906	−19049	75.972	−249580	75.931

Repeating more iterations, the following values are obtained:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.29048 \\ 19.690 \\ 1.0857 \end{bmatrix}$$

When this method is consistent?

Gauss-Seidel Method

If the matrix is strongly diagonally dominant the Gauss-Seidel method is convergent

$$\left|a_{ii}\right| \geq \sum_{j=1, j\neq i}^{n} a_{ij}$$ for all *i*

$$\left|a_{ii}\right| > \sum_{j=1, j\neq i}^{n} a_{ij}$$ for at least one *i*

Example of a matrix diagonally dominant

$$\begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}$$

$$|a_{11}| \geq |a_{12}| + |a_{13}| = 8$$

$$|a_{22}| \geq |a_{21}| + |a_{23}| = 4$$

$$|a_{33}| \geq |a_{31}| + |a_{32}| = 10$$

LU Decomposition

LU Decomposition is another method to solve a set of simultaneous linear equations

$$\mathbf{Ax} = \mathbf{b}$$

The matrix A can be represented as:

$$\mathbf{A} = \mathbf{LU}$$

where:

L – lower triangular matrix

U – upper triangular matrix

LU Decomposition

For a nonsingular matrix [A] on which one can successfully conduct the Naïve Gauss elimination forward elimination steps, one can always write it as:

$$[A] = [L][U]$$

where: $[L]$ =Lower triangular matrix

$[U]$ =Upper triangular matrix

If one is solving a set of equations

$$[A][X] = [C]$$

then:

$$[L][U][X] = [C]$$

LU Decomposition

Multiplying both sides by $[L]^{-1}$ we get

$$[L]^{-1}[L][U][X]=[L]^{-1}[C]$$

but: $[L]^{-1}[L]=[I]$ ← unit matrix

$$[I][U][X]=[L]^{-1}[C]$$

but: $[I][U]=[U]$

therefore: $[U][X]=[L]^{-1}[C]$

LU Decomposition

$$[U][X] = [L]^{-1}[C]$$

You can replace $[U][X] = [Z]$ (2) $[L]^{-1}[C] = [Z]$

$$[L][Z] = [C]$$ (1)

The idea is to solve (2) for [Z] by forward substitution and then to use (2) to calculate the solution vector [X] by back substitution

LU Decomposition

Given:

$$[A][X]=[C]$$

Decompose [***A***] into [***L***] and [***U***]

Solve [*L*][*Z*] = [*C*] for [*Z*]

Solve [*U*][*X*] = [*Z*] for [*X*]

LU Decomposition

[**A**] Decompose to [**L**] and [**U**]

$$[A]=[L][U]=\begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

[*U*] is the same as the coefficient matrix at the end of the forward elimination step.

[*L*] is obtained using the *multipliers* that were used in the forward elimination process

Finding the [*U*] matrix

Using the Forward Elimination Procedure of Gauss Elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Step 1: $$\frac{64}{25} = 2.56; \quad Row2 - Row1(2.56) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix}$$

$$\frac{144}{25} = 5.76; \quad Row3 - Row1(5.76) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

Finding the [U] Matrix

Matrix after Step 1:

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

Step 2: $\frac{-16.8}{-4.8} = 3.5; \quad Row3 - Row2(3.5) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$

$$[U] = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Finding the [L] matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}$$

Using the multipliers used during the Forward Elimination Procedure

From the first step of forward elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

$$\ell_{21} = \frac{a_{21}}{a_{11}} = \frac{64}{25} = 2.56$$

$$\ell_{31} = \frac{a_{31}}{a_{11}} = \frac{144}{25} = 5.76$$

Finding the [L] Matrix

From the second step of forward elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

$$\ell_{32} = \frac{a_{32}'}{a_{22}'} = \frac{-16.8}{-4.8} = 3.5$$

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}$$

Does [L][U] = [A]?

$$[L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} = \textbf{?}$$

Using LU Decomposition to solve SLEs

Solve the following set of linear equations using LU Decomposition

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Using the procedure for finding the [*L*] and [*U*] matrices

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Example

Set $[L][Z] = [C]$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Solve for $[Z]$

$$z_1 = 10$$

$$2.56 z_1 + z_2 = 177.2$$

$$5.76 z_1 + 3.5 z_2 + z_3 = 279.2$$

Example

Complete the forward substitution to solve for [*Z*]

$$z_1 = 106.8$$

$$z_2 = 177.2 - 2.56 z_1$$
$$= 177.2 - 2.56(106.8)$$
$$= -96.2$$

$$z_3 = 279.2 - 5.76 z_1 - 3.5 z_2$$
$$= 279.2 - 5.76(106.8) - 3.5(-96.21)$$
$$= 0.735$$

$$[Z] = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Example

Set $[U][X] = [Z]$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Solve for $[X]$

The 3 equations become

$$25a_1 + 5a_2 + a_3 = 106.8$$

$$-4.8a_2 - 1.56a_3 = -96.21$$

$$0.7a_3 = 0.735$$

Example

From the 3[rd] equation

$$0.7a_3 = 0.735$$

$$a_3 = \frac{0.735}{0.7}$$

$$a_3 = 1.050$$

Substituting in a_3 and using the second equation

$$-4.8a_2 - 1.56a_3 = -96.21$$

$$a_2 = \frac{-96.21 + 1.56a_3}{-4.8}$$

$$a_2 = \frac{-96.21 + 1.56(1.050)}{-4.8}$$

$$a_2 = 19.70$$

Example

Substituting in a_3 and a_2 using the first equation

$$25a_1 + 5a_2 + a_3 = 106.8$$

$$a_1 = \frac{106.8 - 5a_2 - a_3}{25}$$
$$= \frac{106.8 - 5(19.70) - 1.050}{25}$$
$$= 0.2900$$

Hence the Solution Vector is:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.2900 \\ 19.70 \\ 1.050 \end{bmatrix}$$